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Transport equations for chiral fermions to order $\hbar$ and electroweak baryogenesis: Part I

Tomislav Prokopec◇**, Michael G. Schmidt**◇
and Steffen Weinstock•

◇ *Institut für Theoretische Physik, Universität Heidelberg*
Philosophenweg 16, D-69120 Heidelberg, Germany

• *Nuclear Theory Group, Brookhaven National Laboratory, Upton, NY 11973-5000, USA*

Abstract

This is the first in a series of two papers. In this first part, we use the Schwinger-Keldysh formalism to derive semiclassical Boltzmann transport equations, accurate to order $\hbar$, for massive chiral fermions, scalar particles, and for the corresponding CP-conjugate states. Our considerations include complex mass terms and mixing fermion and scalar fields, such that CP-violation is naturally included, rendering the equations particularly suitable for studies of baryogenesis at a first order electroweak phase transition. We provide a quantitative criterion for when the reduction to the diagonal kinetic equations in the mass eigenbasis is justified, leading to a quasiparticle picture even in the case of mixing scalar or fermionic particles. Within the approximations we make, it is possible to first study the Boltzmann equations without the collision term. In a second paper [1] we discuss the collision terms and reduce the Boltzmann equations to fluid equations.

◇`T.Prokopec@, M.G.Schmidt@thphys.uni-heidelberg.de`
•`weinstock@bnl.gov`

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1. INTRODUCTION

The question of how Boltzmann transport equations emerge from quantum field theory is an old problem [2, 3, 4]. It seems that a consensus has been reached, according to which a suitable (loop) truncation of the self-energies in the Kadanoff-Baym equations, supplied by an on-shell reduction, leads to Boltzmann transport equations for the dynamics of quasiparticles in phase space. While procedures leading to classical Boltzmann equations – which formally correspond to taking the limit $\hbar \to 0$ of the Kadanoff-Baym equations – seem to be quite clear and well established [5, 6], no systematic treatment has been pursued beyond the classical approximation, which would lead to a kinetic description valid to linear order in an $\hbar$ expansion (a notable exception is a study of interference between the tree-level and one-loop decay rates of relevance for grand unified baryogenesis [7].) The inclusion of $\hbar$ corrections is absolutely necessary for any treatment of baryogenesis, which represents the primary impetus for the work presented here, since CP-violating effects vanish in the limit when $\hbar \to 0$. More concretely, one deals with particle transport in high temperature relativistic plasmas with Minimal Standard Model particle content, often extended to include exotic, as-of-yet undiscovered new particles, *e.g.* various supersymmetric partners. While $\hbar$ expansions are often identified with loop expansions in field theory, here we associate $\hbar$ primarily to a gradient expansion (which is a field-theoretical generalization of the WKB expansion) with respect to space-time variations of background fields. The relevant examples of background fields for baryogenesis are the mass terms generated by the (Higgs) order parameter at a first order electroweak phase transition; other examples include classical (possibly stochastic) background gauge fields. One important hurdle, which must be overcome in the development of any formalism that aspires to accurately describe transport in plasmas beyond the classical level, is the establishment of a quasiparticle picture. This entails a demonstration of the existence and the identification of the relevant quasiparticle excitations in the plasma, which can be used to encode the relevant dynamical information. This is the problem we solve here. Apart from baryogenesis, potential applications of the formalism developed in this work include propagation of neutrinos in dissipative media, transport of electrons and holes in quantum wires and quantum semiconductor devices, dynamics of Bose-Einstein condensates, *etc.*

The standard approach to electroweak baryogenesis [8] requires a study of the generation and transport of CP-violating flows [9] arising from interactions of fermions with expanding phase transition fronts of a first order electroweak phase transition. The most prominent theoretical problem in such scenarios over the past few years has been to find a systematic derivation of the appropriate transport equations, including the CP-violating sources. In particular, there has been controversy in the literature regarding what are the dominant sources appearing in the transport equations and how to compute them. This work is an attempt to resolve this daunting question.

It is known that a strong first order transition is a necessary requirement for electroweak baryogenesis [10, 11]. While absent in the Minimal Standard Model [12, 13, 14, 15, 16, 17], a strong transition can be realized in its extensions, which include two Higgs doublet models [18], the Minimal Supersymmetric Standard Model (MSSM) [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29], and its nonminimal extensions (NMSSM) [30, 31, 32]. In the NMSSM, not only the possibility of electroweak baryogenesis is given in wide regions of the parameter space, but it can also have stronger CP-violation, increasing the amount of the actually produced baryon asymmetry [33].

Considering the derivation of the relevant transport equations, there are essentially two approaches in literature. The first approach, which we shall refer to as the *semiclassical force mechanism* [34, 35], is based on the observation that, when treated in the WKB approximation, a CP-violating quadratic part of the fermionic Lagrangean exhibits a CP-violating shift at first order in gradients in the dispersion relation, which manifests itself as a CP-violating semiclassical force in the Boltzmann equation. This force then appears in the fluid transport equations, or equivalently diffusion equations for the relevant particle densities. In the original work [34, 35] baryogenesis from scattering of top quarks and tau leptons in two Higgs doublet models was considered. In the subsequent work [36] the semiclassical force formalism was extended to include the case of fermion mixing, which is relevant for example for the *chargino* induced baryogenesis in the Minimal Supersymmetric Standard Model (MSSM) [36, 37, 38, 39, 40] and in its nonminimal extensions (NMSSM) [41, 42].

The second approach, commonly referred to as *spontaneous baryogenesis* [43], is based on the observation that in the presence of CP-violation the (fermion) hypercharge currents are not conserved. As a consequence, the energy levels for CP-conjugate fermionic states are mutually shifted. In the presence of scatterings and transport, this shift then leads to different populations for chiral fermions and the corresponding antifermions, thus sourcing baryogenesis. It was then pointed out in Refs. [44], [35] and [45] that the hypercharge current, which sources spontaneous baryogenesis, is suppressed by the square of the mass. (This must be so simply because the hypercharge current is conserved in the limit when the mass of particles vanishes, so that the spontaneous source must vanish, too.) Mostly in order to incorporate the possibility of fermionic mixing, the spontaneous baryogenesis mechanism was subsequently developed by several groups, which adopted the idea to baryogenesis studies mediated by the stops and charginos of the MSSM [46, 47, 48, 49, 50, 51] and NMSSM [31].

From the current literature, it is in fact not clear whether the semiclassical force and spontaneous sources represent identical or different sources, or if there perhaps has been a conceptual problem in calculating (one or both of) the sources. The situation was made even more controversial by a recent work [51], in which it was claimed that the spontaneous baryogenesis and semiclassical force sources correspond formally to the same source in fluid transport equations, arguing that the method used for computing the semiclassical force source was incorrect.

This is to be compared with the methodology advocated in the work based on the WKB method [39], and more recently in [52, 53, 54, 55] based on the Schwinger-Keldysh formalism [2], or more precisely on a gradient expansion of the Kadanoff-Baym equations [3] in a weak coupling regime. In order to clarify the point of disagreement, we first extend the approach developed in [52, 53]. We consider the dynamics of non-equilibrium Wigner functions for massive chiral fermions which couple to a CP-violating Higgs condensate of advancing phase fronts to first order in a gradient expansion. Assuming that the relevant mass parameter depends only on one spatial coordinate, which models reasonably well the limit of large almost planar bubble walls of a strong first order phase transition, implies a conserved spin, and the constraint equation for the Wigner function is solved by a spectral on-shell *ansatz*, thus proving the validity of the *quasiparticle picture* of the plasma. This result holds both in the case of one fermion and in the case of mixing fermions.

An important difference, regarding the source calculation in spontaneous baryogenesis and semiclassical force baryogenesis, is the choice of the basis in flavor space, which leads to qualitative differences in the resulting baryon production. In studies of chargino-mediated baryogenesis flavor

mixing is of crucial importance. While the proponents of spontaneous baryogenesis argue that the weak interaction (flavor) basis (of charginos) is the right basis to work in, since it is favored by the interactions [51], the semiclassical force camp contends that the mass eigenbasis has apparent advantages. Of course, any physical quantity must be basis independent, but only when one includes flavor mixing in transport equations [56], which – as of yet – has not been done. We strongly favor the mass eigenbasis: as we prove in section 5.2, when working in the mass eigenbasis and to order $\hbar$ accuracy, the diagonal densities decouple from the off-diagonals. The decoupling works uniquely in the mass eigenbasis. The proof is valid provided the mass eigenvalues are not nearly degenerate, that is when $\hbar k\cdot\partial \ll \delta(m_d^2)$ is satisfied, where $\delta(m_d^2)$ signifies the split in the mass eigenvalues. While this strongly indicates that the mass eigenbasis is singled out, the definite resolution of the controversy awaits a basis independent treatment.

The paper is organized as follows.

In section 2 we review the Schwinger-Keldysh formalism, suitable for out-of-equilibrium dynamics of field theoretical correlators, and present a derivation of the relevant Kadanoff-Baym equations in Wigner space for our model Lagrangean of chiral fermions coupled to scalars. We include the case of flavor mixing in both the fermionic and the scalar sector.

In section 3 we study an on-shell reduction of the Kadanoff-Baym equations for a single scalar field. We show that it is consistent to include the (real part of the) self-energy and the collision term to both the constraint and kinetic equation, provided one truncates the self-energy and the collision term at the same order in the coupling constant. This resolves one of the important – and up-to-now unanswered – questions of Boltzmann-like kinetics of relativistic systems. Even though we do not attempt to generalize our result to the kinetics of fermions, it is quite plausible that the same conclusion can be reached. In the subsequent sections we assume that this is the case.

A proof that no source appears in the flow term of the kinetic equations for mixing scalars at order $\hbar$ for planar walls is presented in section 4. We complete the proof originally presented in [52]. Next, we discuss scalar kinetic equations for the CP-conjugate scalar densities, which are then used in Paper II [1] to identify the CP-violating source in the scalar collision term.

Section 5 is devoted to the tree-level kinetics of mixing fermions in the presence of a CP-violating, complex mass matrix. An important application includes chargino and neutralino mediated baryogenesis in supersymmetric extensions of the Minimal Standard Model. We begin by identifying a conserved quantity, which for planar bubble walls and in the wall frame corresponds to the spin in the direction $\vec{n} \propto (k_x k_z, k_y k_z, k_0^2 - k_x^2 - k_y^2)$, which *does not* correspond to helicity, which has been erroneously identified as the conserved quantity in most of the literature on semiclassical force baryogenesis. Then we construct Wigner functions which are block diagonal in the conserved spin. Just like in the scalar case, we prove that to order $\hbar$ the diagonal and off-diagonal densities decouple in the mass eigenbasis (we fill in a gap in the original derivation in Ref. [52]), provided the mass eigenvalues are not nearly degenerate, $\hbar k\cdot\partial \ll \delta(|m_d|^2)$. The final part of the section contains the derivation of Boltzmann-like transport equations for the fermion distribution functions, with particular emphasis on the kinetics of CP-violating densities. The relevant CP-violating sources are identified, some of which are unaccounted for in the existing literature. For example, we show that a CP-even deviation from equilibrium in the distribution function can contribute as a CP-violating source at order $\hbar$.

In Paper II [1] we address the collision terms of the Boltzmann equations, which contain further

CP-violating sources. We furthermore simplify the Boltzmann equations to a set of fluid equations, based on which one can study the dynamics of CP-violating currents.

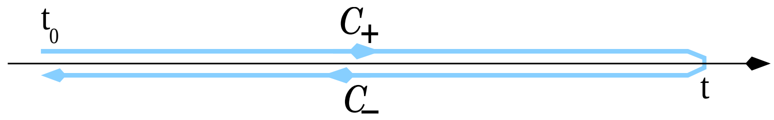

FIG. 1: The complex time contour for the Schwinger-Keldysh non-equilibrium formalism.

2. FROM THE 2PI EFFECTIVE ACTION TO KINETIC THEORY

2.1. Green Functions

The *in-out-formalism* of quantum field theory is well suited for the description of particle scattering experiments, in which the system is prepared to be in a definite *in-state* at $t \to -\infty$ and the question is what is the probability amplitude for the system to be found in a definite *out-state* at $t \to +\infty$. In statistical physics, however, it is often of interest to follow the temporal evolution of a system. Starting with definite initial conditions at some time $t = t_0$, we ask for the expectation values of physical quantities at finite times. A theoretical framework for such problems was first suggested by Schwinger in 1961 [2] and then developed further by Keldysh [4] and others. An extension of field theory capable of dealing with non-equilibrium problems is obtained by defining the time arguments of all quantities on a path $\mathcal{C}$ that leads from the initial time t_0 to t and then back to t_0, as illustrated in figure 1. All integrals and derivatives have then to be performed along that path, and the usual time ordering becomes time ordering $T_\mathcal{C}$ along $\mathcal{C}$. This formalism is called Schwinger-Keldysh formalism or Closed Time Path (CTP) formalism [5]. The definitions of the scalar and the fermionic Green functions are

$$
\begin{aligned}
i\Delta(u,v) &\equiv \langle\Omega|T_C\,\phi(u)\phi^\dagger(v)|\Omega\rangle && (2.1)\\
iS_{\alpha\beta}(u,v) &\equiv \langle\Omega|T_\mathcal{C}\,\psi_\alpha(u)\bar\psi_\beta(v)|\Omega\rangle\,, && (2.2)
\end{aligned}
$$

where $|\Omega\rangle$ denotes the physical state of the system, and ϕ and ψ are the scalar and fermionic fields, respectively. The contour $\mathcal{C}$ can now be split into a $\mathcal{C}_+$ branch from t_0 to t and a $\mathcal{C}_-$ branch from t to t_0, as shown in figure 1. An important simplification, which will help us to perform the Wigner transform, instrumental for gradient expansion, is an extension of the integration path to $t_0 \to -\infty$ and $t \to \infty$. This simplification is suitable for problems of our interest, which comprise plasmas close to chemical and thermal equilibrium with efficient equilibration processes, as it is indeed the case with the electroweak plasma. In this case the influence of initial conditions can be neglected. Denoting the branch on which the time argument lies by an index $a = \pm$, we can rewrite the formalism using ordinary time arguments. We then have

$$
\begin{aligned}
\int_\mathcal{C} d^4u &\to \sum_a a\int_{-\infty}^{\infty} d^4u\\
\delta_\mathcal{C}(u-v) &\to a\delta_{ab}\delta(u-v)\\
i\Delta(u,v) &\to i\Delta^{ab}(u,v)\\
iS(u,v) &\to iS^{ab}(u,v)\,. && (2.3)
\end{aligned}
$$

The additional factors of a in the integral and in the δ-function appear since the $\mathcal{C}_-$ branch runs backward in time.

In this Keldysh formulation one usually defines the following four scalar Green functions:

$$
\begin{aligned}
i\Delta^{++}(u,v) &\equiv i\Delta^{t}(u,v) \;= \langle\Omega|T[\phi(u)\phi^{\dagger}(v)]|\Omega\rangle \\
i\Delta^{+-}(u,v) &\equiv i\Delta^{<}(u,v) = \langle\Omega|\phi^{\dagger}(v)\phi(u)|\Omega\rangle \\
i\Delta^{-+}(u,v) &\equiv i\Delta^{>}(u,v) = \langle\Omega|\phi(u)\phi^{\dagger}(v)|\Omega\rangle \\
i\Delta^{--}(u,v) &\equiv i\Delta^{\bar{t}}(u,v) \;= \langle\Omega|\overline{T}[\phi(u)\phi^{\dagger}(v)]|\Omega\rangle\,,
\end{aligned}
\tag{2.4}
$$

and similarly the fermionic Green functions

$$
\begin{aligned}
iS^{++}_{\alpha\beta}(u,v) &\equiv iS^{t}_{\alpha\beta}(u,v) = \langle\Omega|T[\psi_{\alpha}(u)\bar{\psi}_{\beta}(v)]|\Omega\rangle \\
iS^{+-}_{\alpha\beta}(u,v) &\equiv iS^{<}_{\alpha\beta}(u,v) = -\langle\Omega|\bar{\psi}_{\beta}(v)\psi_{\alpha}(u)|\Omega\rangle \\
iS^{-+}_{\alpha\beta}(u,v) &\equiv iS^{>}_{\alpha\beta}(u,v) = \langle\Omega|\psi_{\alpha}(u)\bar{\psi}_{\beta}(v)|\Omega\rangle \\
iS^{--}_{\alpha\beta}(u,v) &\equiv iS^{\bar{t}}_{\alpha\beta}(u,v) = \langle\Omega|\overline{T}[\psi_{\alpha}(u)\bar{\psi}_{\beta}(v)]|\Omega\rangle\,,
\end{aligned}
\tag{2.5}
$$

where T ($\overline{T}$) denotes time ('anti-time') ordering and the additional minus sign in the second fermion line is due to the anticommutation property of the fermionic fields.

These definitions imply immediately the following hermiticity properties for the Wightman functions

$$
\left(i\Delta^{<,>}(u,v)\right)^{\dagger} = i\Delta^{<,>}(v,u) \tag{2.6}
$$

$$
\left(i\gamma^{0}S^{<,>}(u,v)\right)^{\dagger} = i\gamma^{0}S^{<,>}(v,u)\,. \tag{2.7}
$$

The four two-point functions (2.4) and (2.5) are not completely independent. Indeed, with the definition of time and anti-time ordering one finds for the chronological (Feynman) and anti-chronological Green functions

$$
\begin{aligned}
G^{t}(u,v) &= \theta(u_0-v_0)G^{>}(u,v)+\theta(v_0-u_0)G^{<}(u,v) \\
G^{\bar{t}}(u,v) &= \theta(u_0-v_0)G^{<}(u,v)+\theta(v_0-u_0)G^{>}(u,v)\,,
\end{aligned}
\tag{2.8}
$$

where we have used the generic notation $G=\{\Delta,S\}$. From now on we use this notation in relations which are identical for bosonic and fermionic Green functions.

From the definitions of the retarded and advanced propagators

$$
\begin{aligned}
G^{r} &\equiv G^{t}-G^{<} = G^{>}-G^{\bar{t}} \\
G^{a} &\equiv G^{t}-G^{>} = G^{<}-G^{\bar{t}}
\end{aligned}
\tag{2.9}
$$

and the definitions of their hermitean and antihermitean parts

$$
\begin{aligned}
G_{h} &\equiv \frac{1}{2}(G^{r}+G^{a}) \\
G_{a} &\equiv \frac{1}{2i}(G^{a}-G^{r}) = \frac{i}{2}(G^{>}-G^{<}) \equiv \mathcal{A}\,,
\end{aligned}
\tag{2.10}
$$

one obtains the spectral relation

$$
G_{h}(u,v) = -i\,\mathrm{sign}(u^{0}-v^{0})\,\mathcal{A}(u,v)\,, \tag{2.11}
$$

where $\mathrm{sign}(u^0-v^0)=\Theta(u^0-v^0)-\Theta(v^0-u^0)$, $u^{\mu}=(u^0,\vec{u})$. $\mathcal{A}$ is called the spectral function.

2.2. Lagrange density

We shall now consider the dynamics of fermionic and scalar particles in the presence of a scalar condensate which may give space-time dependent masses to both types of particles, and we shall assume that scalars and fermions couple *via* a Yukawa interaction term. The tree-level action is then

$$
\begin{aligned}
I[\phi, \psi] &= \int_{\mathcal{C}} d^4u\, \mathcal{L} &\quad (2.12)\\
\mathcal{L} &= i\bar{\psi}\, \partial\!\!\!/\, \psi - \bar{\psi}_L m \psi_R - \bar{\psi}_R m^* \psi_L + (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \phi^\dagger M^2 \phi + \mathcal{L}_{\rm int}\,, &\quad (2.13)
\end{aligned}
$$

where the Yukawa interaction Lagrangean reads

$$
\begin{aligned}
\mathcal{L}_{\rm int} &= -\bar{\psi}_L y\phi\psi_R - \bar{\psi}_R (y\phi)^\dagger \psi_L \\
&= -\bar{\psi}\left(P_R \otimes y\phi + P_L \otimes (y\phi)^\dagger\right)\psi\,, \qquad (2.14)
\end{aligned}
$$

and we can rewrite the fermionic mass term as

$$
-\bar{\psi}_L m \psi_R - \bar{\psi}_R m^* \psi_L = -\bar{\psi}(\mathbb{1} \otimes m_h + i\gamma^5 \otimes m_a)\psi\,. \qquad (2.15)
$$

Fermionic fields with definite chirality $\psi_{R,L} = P_{R,L}\psi$ are defined with the help of the chirality projection operator $P_{R,L} = (\mathbb{1} \pm \gamma^5)/2$, and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Our analysis includes the possibility of particle flavor mixing, but for notational simplicity we have suppressed scalar and fermionic flavor labels (we shall explicitly state so when we consider the special case of the single field dynamics without mixing.) Thus, ϕ and ψ are vectors in the scalar and fermionic flavor space, respectively, and y is a vector in the scalar and a matrix in the fermionic flavor space. Then $\otimes$ denotes a direct product of the spinor and fermionic flavor spaces. M and m are complex matrices in the scalar and fermionic flavor space, respectively, whose non-diagonal elements couple fields of different flavors to each other. Even though the Lagrangean (2.13) cannot fully describe the physics of the Standard Model, since it does not contain gauge fields, it contains many essential elements of the physics of CP-violation and interactions of importance for baryogenesis. Since the main purpose of this work is to establish controlled calculational methods, the Lagrangean (2.13) should be a good toy model for studying transport, in particular for baryogenesis.

An important example of the scalar field condensate is the Higgs field at a first order electroweak phase transition in variants of the standard model. This phase transition proceeds via nucleation and growth of the broken phase bubbles of a nonzero Higgs condensate, which varies at the interface of the symmetric and the broken phases, the so called bubble wall. The phase transition bubbles quickly grow large in comparison with the thickness of the phase transition front, so for studies of local transport phenomena at the scale of the bubble walls one can to a good approximation assume planar symmetry. We will explicitly make use of this symmetry in later sections. To allow for CP-violation we choose the fermion mass to be a complex matrix

$$
m(u) = m_h(u) + i m_a(u) = |m(u)| \mathrm{e}^{i\theta(u)}\,, \qquad (2.16)
$$

where m_h and m_a denote the hermitean and antihermitean part of m, respectively. The last part of this equation is to be understood as an equation for the components. CP-violation can be mediated either through m_a or complex off-diagonal entries of m_h. The scalar mass matrix is by construction hermitean, ${M^2}^\dagger = M^2$. Nevertheless, in the case of flavor mixing CP-violation can be mediated through the off-diagonal elements of M^2, provided they are complex.

2.3. Two-particle-irreducible (2PI) effective action

We shall now review how to derive the equation of motion for the Green functions in the CTP-formalism. Note first that the tree-level equation can be obtained in a straightforward manner as follows. By varying the action (2.12) with respect to $\phi^\dagger$, one obtains the familiar Klein-Gordon equation

$$\big(-\partial_u^2 - M^2(u)\big)\phi(u) = 0\,. \tag{2.17}$$

After multiplying this from the left by $\phi^\dagger(v)$ and taking the expectation value one gets the tree-level equation of motion for the Wightman function (2.4):

$$\big(-\partial_u^2 - M^2(u)\big)i\Delta^{<}(u,v) = 0\,. \tag{2.18}$$

By an analogous procedure one finds that the same equation holds for $i\Delta^{>}$. The equations of motion for $i\Delta^t$ and $i\Delta^{\bar t}$ are obtained by imposing the appropriate time ordering, such that they acquire on the *r.h.s.* the δ-function sources $i\delta^4(u-v)$ and $-i\delta^4(u-v)$, respectively. Similarly one finds that at tree-level the fermionic Wightman functions (2.5) obey the Dirac equation

$$\big(i\,\partial\!\!\!/_u - m_h(u) - i\gamma^5 m_a(u)\big)\, iS^{<,>}(u,v) = 0, \tag{2.19}$$

and the equations of motion for iS^t and $iS^{\bar t}$ again acquire $i\delta^4(u-v)$ and $-i\delta^4(u-v)$ on the *r.h.s.*, respectively.

In order to include interactions into the equations of motion, we use the 2PI effective action approach, which was originally developed for equilibrium problems in condensed matter physics (where it is better known as Φ-derivable actions) by Luttinger and Ward [57] and in relativistic field theory by Cornwall, Jackiw and Tomboulis [58] and then adapted to non-equilibrium situations by Chou, Su, Hao and Yu [5], and by Calzetta and Hu [6]. In this approach one uses the two-particle irreducible (2PI) out-of-equilibrium formulation for the CTP effective action Γ, which is a functional not only of the expectation value of the field $\varphi(u) \equiv \langle\Omega|\phi(u)|\Omega\rangle$, but also of the two-point functions $i\Delta(u,v)$ and $iS(u,v)$ defined in (2.1-2.2). The main advantage of this formalism compared to the more standard 1PI approach is that, at a given order in the loop expansion, varying the 2PI action probes much more accurately the field configuration space. Technically speaking, at a given order in the loop expansion the 2PI effective action typically resums a much larger set of diagrams than the corresponding 1PI effective action. In absence of sources, the equations of motion are obtained simply by extremising the effective action with respect to $\varphi(u)$, $\Delta(u,v)$ and $S(u,v)$:

$$\begin{aligned}
\frac{\delta\Gamma[\varphi,\Delta,S]}{\delta\varphi} &= 0\\
\frac{\delta\Gamma[\varphi,\Delta,S]}{\delta\Delta} &= 0\\
\frac{\delta\Gamma[\varphi,\Delta,S]}{\delta S} &= 0\,.
\end{aligned} \tag{2.20}$$

These equations are obtained by the variation of the 2PI effective action, and hence they correspond to the Schwinger-Dyson equations, with the self-energy approximated by the 1PI diagrams. Since we are here not interested in modeling the dynamics of the scalar field condensate, we shall not

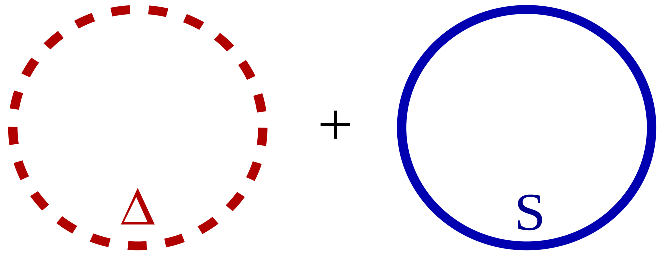

FIG. 2: The one-loop diagrams contributing to the 2PI effective action (2.21).

study the equation of motion for φ. For our purposes we consider φ as given and simply absorb the effects of φ into space-time dependent mass terms m and M^2. The quantum dynamics of a self-interacting scalar field in the presence of a condensate describing the inflaton decay is considered in the framework of the 2PI effective action approach for example in [59, 60, 61], while the equivalent problem for classical scalar fields is studied in [62, 63]. Various aspects of the out-of-equilibrium dynamics and thermalization of quantum fields are considered in [64, 65, 66, 67]. Here we develop a gradient approximation for the two-point function, starting with the 2PI effective action, which is appropriate for studying the dynamics of fields in the presence of slowly varying backgrounds.

The 2PI effective action corresponding to the classical action (2.12) can be written in the form [58]

$$\Gamma[\Delta, S] = i\mathrm{Tr}({\Delta^{(0)}}^{-1}\Delta) - i\mathrm{Tr}({S^{(0)}}^{-1}S) + i\mathrm{Tr}\ln\Delta^{-1} - i\mathrm{Tr}\ln S^{-1} + \Gamma_2[\Delta, S]\,, \tag{2.21}$$

where the minus signs in the fermionic terms are related to Pauli statistics. We use a condensed notation, with Tr denoting both integration over space-time and summation over spinor and flavor indices. The first two terms in (2.21) are the classical (tree-level) actions, the next two terms correspond to the one-loop vacuum diagrams shown in figure 2, while the last term Γ_2 stands for the sum of all two-particle irreducible vacuum graphs. In Paper II we illustrate how to compute Γ_2 in practice. The inverse free propagators ${\Delta^{(0)}}^{-1}$ and ${S^{(0)}}^{-1}$ can be read off from the classical action (2.12-2.13) rewritten as

$$I[\phi, \psi] = \int_{\mathcal{C}} d^4u\, d^4v\, \phi^\dagger(u){\Delta^{(0)}}^{-1}(u,v)\phi(v) + \int_{\mathcal{C}} d^4u\, d^4v\, \bar\psi(u){S^{(0)}}^{-1}(u,v)\psi(v) + \int_{\mathcal{C}} d^4u\mathcal{L}_{\mathrm{int}}\,. \tag{2.22}$$

They satisfy equations (2.18-2.19) and are given by

$$\begin{aligned} {\Delta^{(0)}}^{-1}(u,v) &= \big(-\partial_u^2 - M^2(u)\big)\delta^4_{\mathcal{C}}(u-v) \\ {S^{(0)}}^{-1}(u,v) &= \big(i\,\partial\!\!\!/_u - m_h - i\gamma^5 m_a\big)\,\delta^4_{\mathcal{C}}(u-v)\,. \end{aligned} \tag{2.23}$$

We now take the functional derivatives of the effective action (2.21) with respect to Δ and S to obtain

$$\begin{aligned} \frac{\delta\Gamma[\Delta,S]}{\delta\Delta(v,u)} &= i{\Delta^{(0)}}^{-1}(u,v) - i\Delta^{-1}(u,v) + \frac{\delta\Gamma_2[\Delta,S]}{\delta\Delta(v,u)} = 0 \\ \frac{\delta\Gamma[\Delta,S]}{\delta S(v,u)} &= -i{S^{(0)}}^{-1}(u,v) + iS^{-1}(u,v) + \frac{\delta\Gamma_2[\Delta,S]}{\delta S(v,u)} = 0\,. \end{aligned} \tag{2.24}$$

By making use of the definitions for the scalar and fermionic self-energies

$$\Pi(u,v) \equiv i\frac{\delta\Gamma_2[\Delta,S]}{\delta\Delta(v,u)} \tag{2.25}$$

$$\Sigma(u,v) \equiv -i\frac{\delta\Gamma_2[\Delta,S]}{\delta S(v,u)} \tag{2.26}$$

and multiplying from the right by Δ and $-S$, respectively, we can recast (2.24) as the Schwinger-Dyson equations

$$\big(-\partial_u^2 - M^2(u)\big)i\Delta(u,v) = i\delta^4_{\mathcal{C}}(u-v) + \int_{\mathcal{C}} d^4w\,\Pi(u,w)i\Delta(w,v) \tag{2.27}$$

$$\big(i\,\partial\!\!\!/_u - m_h(u) - i\gamma^5 m_a(u)\big)iS(u,v) = i\delta^4_{\mathcal{C}}(u-v) + \int_{\mathcal{C}} d^4w\,\Sigma(u,w)iS(w,v)\,. \tag{2.28}$$

So far we have written everything in the complex contour notation. In the index notation the self-energies are

$$\Pi^{ab}(u,v) = iab\frac{\delta\Gamma_2[\Delta,S]}{\delta\Delta^{ba}(v,u)} \tag{2.29}$$

$$\Sigma^{ab}(u,v) = -iab\frac{\delta\Gamma_2[\Delta,S]}{\delta S^{ba}(v,u)}\,, \tag{2.30}$$

and the equations of motion (2.27-2.28) read

$$\big(-\partial_u^2 - M^2(u)\big)i\Delta^{ab}(u,v) = ai\delta_{ab}\delta^4(u-v) + \sum_c c\int d^4w\,\Pi^{ac}(u,w)i\Delta^{cb}(w,v) \tag{2.31}$$

$$\big(i\,\partial\!\!\!/_u - m_h(u) - i\gamma^5 m_a(u)\big)iS^{ab}(u,v) = ai\delta_{ab}\delta^4(u-v) + \sum_c c\int d^4w\,\Sigma^{ac}(u,w)iS^{cb}(w,v)\,. \tag{2.32}$$

These are the fundamental quantum dynamical equations corresponding to the action (2.12-2.13). They look deceptively simple, since the full complexity of the problem is hidden in the self-energies , which are very complicated functionals of the Green functions and in general not known completely.

From Eqs. (2.4-2.5) and (2.31-2.32) we infer the bosonic equations

$$(-\partial^2 - M^2)\Delta^{r,a} - \Pi^{r,a}\odot\Delta^{r,a} = \delta \tag{2.33}$$

$$(-\partial^2 - M^2)\Delta^{<,>} - \Pi^{r}\odot\Delta^{<,>} = \Pi^{<,>}\odot\Delta^{a}\,, \tag{2.34}$$

and the fermionic equations

$$(i\partial\!\!\!/ - m_h - i\gamma^5 m_a)S^{r,a} - \Sigma^{r,a}\odot S^{r,a} = \delta \tag{2.35}$$

$$(i\partial\!\!\!/ - m_h - i\gamma^5 m_a)S^{<,>} - \Sigma^{r}\odot S^{<,>} = \Sigma^{<,>}\odot S^{a}\,, \tag{2.36}$$

where $\odot$ stands for integration over the intermediate variable. We use the notation $\Pi^t \equiv \Pi^{++}$, *etc.* (*cf.* Eqs. (2.4-2.5)) and have defined retarded and advanced self-energies in analogy with (2.9). As we have already pointed out, not all Green functions are independent: G^r and G^a can be expressed in terms of $G^<$ and $G^>$ (*cf.* Eqs. (2.8-2.9)), implying that the equations for the retarded and advanced

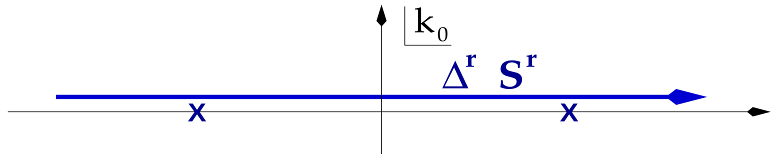

FIG. 3: Integration contour for the retarded propagators Δ^r and S^r in (2.38-2.39).

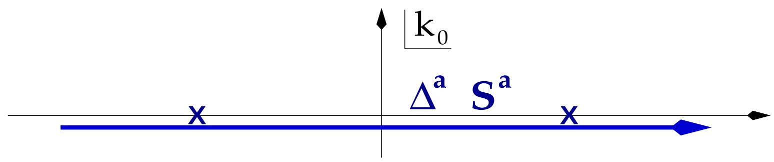

FIG. 4: Integration contour for the advanced propagators Δ^a and S^a in (2.38-2.39).

propagators are redundant. It is not difficult to show that Eqs. (2.33-2.36) are consistent provided the relations

$$\begin{aligned}\Pi^t(u,v) &= \delta^4(u-v)\Pi^{\mathrm{sg}}(u) + \theta(u_0-v_0)\Pi^>(u,v) + \theta(v_0-u_0)\Pi^<(u,v)\\ \Pi^{\bar t}(u,v) &= \delta^4(u-v)\Pi^{\mathrm{sg}}(u) + \theta(u_0-v_0)\Pi^<(u,v) + \theta(v_0-u_0)\Pi^>(u,v)\,,\end{aligned} \tag{2.37}$$

as well as corresponding relations for the fermionic self-energies, are satisfied. This should be the case for any reasonable approximation of the self-energies. The additional singular terms $\Pi^{\mathrm{sg}}(u)$ and $\Sigma^{\mathrm{sg}}(u)$ we have allowed for appear naturally in some theories and can be absorbed by the mass terms (mass renormalization). An example of a singular self-energy contribution is the one-loop tadpole of a scalar theory with quartic self-interaction. The one-loop expressions for the self-energies that we will derive in Paper II indeed satisfy (2.37) and the corresponding fermionic equations.

To clarify the physical meaning of Eqs. (2.33-2.36), we observe that the equations for the retarded and advanced propagators (2.33) and (2.35) describe mostly the spectral properties of the system. Indeed, this can be seen for example from the tree-level solutions, which in the limit of constant mass terms and no flavor mixing read

$$\Delta_0^{r,a}(u,v) = \int \frac{d^4k}{(2\pi)^4} \mathrm{e}^{-ik\cdot(u-v)} \frac{1}{k^2 - M^2 \pm i\mathrm{sign}(k_0)\epsilon} \tag{2.38}$$

$$S_0^{r,a}(u,v) = \int \frac{d^4k}{(2\pi)^4} \mathrm{e}^{-ik\cdot(u-v)} \frac{\not{k} + m_h - i\gamma^5 m_a}{k^2 - m_h^2 - m_a^2 \pm i\mathrm{sign}(k_0)\epsilon}\,, \tag{2.39}$$

where we use the infinitesimal pole shifts $\mp i\epsilon$ ($\epsilon \to 0+$) to indicate the standard integration prescription for the retarded and advanced propagators, respectively, shown in figures 3 and 4. The spectral functions (2.10) are then

$$\mathcal{A}_{\phi 0} = \int \frac{d^4k}{(2\pi)^4} \mathrm{e}^{-ik\cdot(u-v)} \pi\mathrm{sign}(k_0)\delta(k^2 - M^2) \tag{2.40}$$

$$\mathcal{A}_{\psi 0} = \int \frac{d^4k}{(2\pi)^4} \mathrm{e}^{-ik\cdot(u-v)} (\not{k} + m_h - i\gamma^5 m_a)\pi\mathrm{sign}(k_0)\delta(k^2 - m_h^2 - m_a^2)\,. \tag{2.41}$$

The δ-functions indicate that the frequencies of the plasma excitations are constrained to lie on the mass shell, which for scalars is given by

$$k_0 = \pm\omega_\phi\,, \qquad \omega_\phi = \sqrt{\vec{k}^2 + M^2} \tag{2.42}$$

and for fermions by

$$k_0 = \pm\omega_0\,, \qquad \omega_0 = \sqrt{\vec{k}^2 + m_h^2 + m_a^2}\,. \tag{2.43}$$

On the other hand, the equations of motion for the Wightman functions (2.34) and (2.36) describe mostly statistical (kinetic) properties of the system. This can be seen for example from considering the thermal equilibrium solutions to these equations, which are given in section 2.5 below. The kinetic equations (2.34) and (2.36) can be rewritten in the form of the Kadanoff-Baym (KB) equations

$$(-\partial^2 - M^2)\Delta^{<,>} - \Pi_h \odot \Delta^{<,>} - \Pi^{<,>} \odot \Delta_h \;=\; \mathcal{C}_\phi \equiv \frac{1}{2}(\Pi^> \odot \Delta^< - \Pi^< \odot \Delta^>) \tag{2.44}$$

$$(i\partial\!\!\!/ - m_h - i\gamma^5 m_a)S^{<,>} - \Sigma_h \odot S^{<,>} - \Sigma^{<,>} \odot S_h \;=\; \mathcal{C}_\psi \equiv \frac{1}{2}(\Sigma^> \odot S^< - \Sigma^< \odot S^>)\,, \tag{2.45}$$

where $\mathcal{C}_\phi$ and $\mathcal{C}_\psi$ denote the scalar and fermionic collision terms, respectively, and the hermitean parts of the self-energies are defined analogously to those for the Wightman functions (2.10). With reasonable approximations for the self-energies, the Kadanoff-Baym equations are suited to study kinetics of out-of-equilibrium quantum systems. The terms $\Pi_h\Delta^<$ and $\Sigma_h S^<$ on the *l.h.s.* of (2.44-2.45) represent the self-energy contributions, which are sometimes considered as a (nonlocal) contribution to the mass terms. We postpone the discussion of how inclusion of the self-energy may change the plasma dynamics and in particular the quasiparticle picture to a future work. The collision terms $\mathcal{C}_\phi$ and $\mathcal{C}_\psi$, as we will see in Paper II, contain the standard gain and loss terms responsible for equilibration, but they may also contain additional CP-violating sources. Finally, the terms $\Pi^<\Delta_h$ and $\Sigma^< S_h$ induce broadening of the on-shell dispersion relations and may cause breakdown of the quasiparticle picture. We shall consider the role of these terms in some detail in section 3. There we will need the width

$$\Gamma_\phi = \frac{1}{2i}(\Pi^a - \Pi^r) = \frac{i}{2}(\Pi^> - \Pi^<) \tag{2.46}$$

and the hermitean part of the scalar self-energy

$$\Pi_h = \frac{1}{2}(\Pi^r + \Pi^a) = \Pi^t - \frac{1}{2}(\Pi^> + \Pi^<)\,, \tag{2.47}$$

which satisfy the spectral relation (*cf.* Eq. (2.11))

$$\Pi_h(u,v) = \frac{1}{2}\mathrm{sign}(u^0 - v^0)\,[\Pi^>(u,v) - \Pi^<(u,v)] = -i\mathrm{sign}(u^0 - v^0)\,\Gamma_\phi(u,v)\,. \tag{2.48}$$

Analogous relations hold for the fermionic self-energies.

2.4. Wigner representation and gradient expansion

In equilibrium the Green functions $\Delta(u,v)$ and $S(u,v)$ depend only on the relative (microscopic) coordinate $r = u - v$. This dependence corresponds to the internal fluctuations that typically take place on microscopic scales. In non-equilibrium situations, however, Δ and S depend also on the average (macroscopic) coordinate $x = (u+v)/2$. This dependence describes the system's behavior on large, macroscopic scales. For example, if the system couples to an external field which varies in space and time, the Green functions will show a corresponding dependence on the average coordinate.

Thick bubble walls at a first order electroweak phase transition represent precisely an example of such an external field.

In order to separate the microscopic scale fluctuations from the behavior on macroscopical scales one typically performs a Wigner transform, which is a Fourier transform with respect to the relative coordinate r. The Green function in the Wigner representation, called Wigner function, is

$$G(k,x) = \int d^4r\, \mathrm{e}^{ik\cdot r}\, G(x+r/2, x-r/2)\,. \tag{2.49}$$

In the Wigner representation the hermiticity properties (2.6) and (2.7) become simply

$$\begin{aligned} \big(i\Delta^{<,>}(k,x)\big)^\dagger &= i\Delta^{<,>}(k,x) & (2.50)\\ \big(i\gamma^0 S^{<,>}(k,x)\big)^\dagger &= i\gamma^0 S^{<,>}(k,x)\,. & (2.51)\end{aligned}$$

In order to transform the equations of motion, we make use of the general relation

$$\int d^4(u-v)\, \mathrm{e}^{ik\cdot(u-v)} \int d^4w A(u,w)B(w,v) = \mathrm{e}^{-i\diamond}\{A(k,x)\}\{B(k,x)\}\,, \tag{2.52}$$

where x is the average coordinate and the diamond operator is defined by

$$\diamond\{\cdot\}\{\cdot\} = \frac{1}{2}\big(\partial^{(1)}\cdot\partial_k^{(2)} - \partial_k^{(1)}\cdot\partial^{(2)}\big)\{\cdot\}\{\cdot\}\,. \tag{2.53}$$

The superscripts (1) and (2) refer to the first and second argument, respectively, and $\partial \equiv \partial_x$. Since the nonlocal terms have the form of (2.52), it is a simple exercise to show that the Kadanoff-Baym equations (2.44-2.45), when written in the Wigner representation, become

$$\begin{aligned} \Big(k^2 - \frac{1}{4}\partial^2 + ik\cdot\partial - M^2 \mathrm{e}^{-\frac{i}{2}\overleftarrow{\partial}\cdot\partial_k}\Big)\Delta^{<,>} - \mathrm{e}^{-i\diamond}\{\Pi_h\}\{\Delta^{<,>}\} - \mathrm{e}^{-i\diamond}\{\Pi^{<,>}\}\{\Delta_h\} &= \mathcal{C}_\phi & (2.54)\\ \Big(\not k + \frac{i}{2}\not\partial - m_h \mathrm{e}^{-\frac{i}{2}\overleftarrow{\partial}\cdot\partial_k} - i\gamma^5 m_a \mathrm{e}^{-\frac{i}{2}\overleftarrow{\partial}\cdot\partial_k}\Big)S^{<,>} - \mathrm{e}^{-i\diamond}\{\Sigma_h\}\{S^{<,>}\} - \mathrm{e}^{-i\diamond}\{\Sigma^{<,>}\}\{S_h\} &= \mathcal{C}_\psi\,, & (2.55)\end{aligned}$$

where the collision terms are

$$\begin{aligned} \mathcal{C}_\phi &= \frac{1}{2}\mathrm{e}^{-i\diamond}\big(\{\Pi^>\}\{\Delta^<\} - \{\Pi^<\}\{\Delta^>\}\big) & (2.56)\\ \mathcal{C}_\psi &= \frac{1}{2}\mathrm{e}^{-i\diamond}\big(\{\Sigma^>\}\{S^<\} - \{\Sigma^<\}\{S^>\}\big)\,. & (2.57)\end{aligned}$$

We shall analyze these equations in gradient expansion: we assume that the variation of the background field, and therefore also the variation of the Green functions and self-energies, with respect to the macroscopic coordinate x is small when compared with the momenta of plasma excitations. Then we can perform an expansion in derivatives with respect to x. Formally, this criterion reads

$$\partial \ll k. \tag{2.58}$$

That is, the background field is assumed to have a characteristic length scale that is large in comparison to the de Broglie wavelength of the particles in the plasma.

This assumption of a slowly varying background field seems to be justified in our case [68]: in the MSSM, for instance, the width of the bubble wall L_w is roughly $10/T$, where T is the temperature

of the plasma. The typical momentum of a particle in the plasma is of the order T, so that the de Broglie wavelength $l_{dB} \sim 1/T$ is indeed small when compared to L_w. Since the expansion in powers of derivatives can be viewed as an expansion in powers of the Planck constant $\hbar$, the gradient expansion is equivalent to a semiclassical expansion. We expect that the leading order terms in the gradient expansion correspond to classical kinetic equations, while higher order derivatives represent quantum corrections.

2.5. Sum rules and Equilibrium Green functions

It is easily shown that, as a consequence of the equal-time commutation relation for scalars, the Wigner transform of their spectral function $\mathcal{A}_\phi$ (2.10) obeys the spectral sum rule

$$\int_{-\infty}^{\infty} \frac{dk_0}{\pi} k_0 \mathcal{A}_\phi(k,x) = 1\,, \tag{2.59}$$

while the anticommutation rule for fermions leads to

$$\int_{-\infty}^{\infty} \frac{dk_0}{\pi} \gamma^0 \mathcal{A}_\psi(k,x) = \mathbb{1} \tag{2.60}$$

for the fermionic spectral function. These sum rules are important, since they have to be imposed as consistency conditions on the solutions for the Wigner functions $G^<$ and $G^>$, or for G^a and G^r, as indicated in (2.10). Furthermore, making use of these spectral sum rules and the Kubo-Martin-Schwinger (KMS) relations

$$\begin{aligned} \Delta^>_{\rm eq}(k) &= \mathrm{e}^{\beta k_0}\Delta^<_{\rm eq}(k) \\ S^>_{\rm eq}(k) &= -\mathrm{e}^{\beta k_0} S^<_{\rm eq}(k)\,, \end{aligned} \tag{2.61}$$

where $\beta = 1/T$ denotes the inverse temperature, one can unambiguously obtain the thermal equilibrium propagators [69]. They are translationally invariant in space and time and hence, when written in the Fourier space, defined by the Fourier transform with respect to the relative coordinate $r = u - v$, they display a dependence on the momentum only:

$$G_{\rm eq}(k) = \int d^4r\, \mathrm{e}^{ik\cdot r} G_{\rm eq}(r)\,. \tag{2.62}$$

The bosonic and fermionic equilibrium propagators are the solutions of the Klein-Gordon (2.18) and the Dirac equation (2.19), respectively, which in the Fourier space read

$$\begin{aligned} \big(k^2 - M^2\big) i\Delta^{<,>}_{\rm eq}(k) &= 0\,, & \big(k^2 - M^2\big) i\Delta^{r,a}_{\rm eq}(k) &= i \\ \big(\not{k} - m_h - i\gamma^5 m_a\big)\, iS^{<,>}_{\rm eq}(k) &= 0\,, & \big(\not{k} - m_h - i\gamma^5 m_a\big)\, iS^{r,a}_{\rm eq}(k) &= i\,, \end{aligned} \tag{2.63}$$

with the normalization (2.59-2.60) and the boundary conditions (2.61). The scalar equilibrium Green functions are

$$i\Delta^t_{\rm eq}(k) = \frac{i}{k^2 - M^2 + i\mathrm{sign}(k_0)\epsilon} + 2\pi\delta(k^2 - M^2)\mathrm{sign}(k_0) n^\phi_{\rm eq}(k_0) \tag{2.64}$$

$$i\Delta^{\bar t}_{\rm eq}(k) = -\frac{i}{k^2 - M^2 + i\mathrm{sign}(k_0)\epsilon} + 2\pi\delta(k^2 - M^2)\mathrm{sign}(k_0)(1 + n^\phi_{\rm eq}(k_0)) \tag{2.65}$$

$$i\Delta^<_{\rm eq}(k) = 2\pi\delta(k^2 - M^2)\mathrm{sign}(k_0) n^\phi_{\rm eq}(k_0) \tag{2.66}$$

$$i\Delta^>_{\rm eq}(k) = 2\pi\delta(k^2 - M^2)\mathrm{sign}(k_0)(1 + n^\phi_{\rm eq}(k_0))\,, \tag{2.67}$$

where the Bose-Einstein distribution appears:

$$n_{\rm eq}^{\phi}(k_0) = \frac{1}{{\rm e}^{\beta k_0} - 1}\,. \tag{2.68}$$

Similarly, for the fermions we have

$$\begin{aligned}
iS_{\rm eq}^{t}(k) &= \frac{i(\not{k} + m_h - i\gamma^5 m_a)}{k^2 - |m|^2 + i{\rm sign}(k_0)\epsilon} - 2\pi(\not{k} + m_h - i\gamma^5 m_a)\delta(k^2 - |m|^2){\rm sign}(k_0)n_{\rm eq}(k_0) && (2.69)\\
iS_{\rm eq}^{\bar{t}}(k) &= -\frac{i(\not{k} + m_h - i\gamma^5 m_a)}{k^2 - |m|^2 + i{\rm sign}(k_0)\epsilon} + 2\pi(\not{k} + m_h - i\gamma^5 m_a)\delta(k^2 - |m|^2){\rm sign}(k_0)(1 - n_{\rm eq}(k_0)) && (2.70)\\
iS_{\rm eq}^{<}(k) &= -2\pi(\not{k} + m_h - i\gamma^5 m_a)\delta(k^2 - |m|^2){\rm sign}(k_0)n_{\rm eq}(k_0) && (2.71)\\
iS_{\rm eq}^{>}(k) &= 2\pi(\not{k} + m_h - i\gamma^5 m_a)\delta(k^2 - |m|^2){\rm sign}(k_0)(1 - n_{\rm eq}(k_0))\,, && (2.72)
\end{aligned}$$

with the Fermi-Dirac distribution function

$$n_{\rm eq}(k_0) = \frac{1}{{\rm e}^{\beta k_0} + 1} \tag{2.73}$$

and $|m|^2 \equiv m_h^2 + m_a^2$. For the case of several mixing particle species, the equilibrium Green functions are diagonal matrices in flavor space, and the given relations hold for the diagonal elements.

3. REDUCTION TO THE ON-SHELL LIMIT

In this section [79] we pay particular attention to the self-consistency of the on-shell limit as a controlled expansion in the coupling constants of the theory. To this aim we consider the role of the dissipative terms on the *l.h.s.* of the propagator equations (2.33) and the Kadanoff-Baym equation (2.54). We shall see that including these terms into the retarded and advanced propagators results in a Breit-Wigner form for the spectral function. By making use of the spectral decomposition ansatz for the Wigner function with the Breit-Wigner form for the spectral function, we shall study under which conditions the kinetic equation reduces to the on-shell form.

For simplicity, we consider here the scalar case. It is quite plausible that similar techniques can be used to extend the analysis presented in this section to the case of fermions. Let us begin by rewriting equations (2.33) and (2.54) more compactly,

$$\mathrm{e}^{-i\diamond}\{\Omega_\phi^2 \pm i\Gamma_\phi\}\{\Delta^{r,a}\} \;=\; 1 \tag{3.1}$$

$$\mathrm{e}^{-i\diamond}\{\Omega_\phi^2\}\{\Delta^{<,>}\} - \mathrm{e}^{-i\diamond}\{\Pi^{<,>}\}\{\Delta_h\} \;=\; \mathcal{C}_\phi\,, \tag{3.2}$$

where $\mathcal{C}_\phi$ is defined in Eq. (2.56), and we defined

$$\Omega_\phi^2 \;=\; k^2 - M^2 - \Pi_h\,. \tag{3.3}$$

Upon adding and subtracting the complex conjugate of (3.1), we arrive at the following propagator equations for the case of a single scalar field up to second order in gradients

$$\cos\diamond\big\{\Omega_\phi^2 \pm i\Gamma_\phi\big\}\big\{\Delta^{r,a}\big\} \;=\; 1 \tag{3.4}$$

$$\sin\diamond\big\{\Omega_\phi^2 \pm i\Gamma_\phi\big\}\big\{\Delta^{r,a}\big\} \;=\; 0, \tag{3.5}$$

where we used the fact that $\Delta^{r,a}$ commute with Ω_ϕ^2 and Γ_ϕ. This commutation property is special to the single field case however. Indeed, in the case of mixing scalars, $\Delta^{r,a}$ are matrices which in general do not commute with Ω_ϕ^2 and Γ_ϕ.

Now observe that Eq. (3.5) can be obtained from Eq. (3.4) by an application of the differential operator $\tan\diamond$, and hence it gives no new information. Because $\cos\diamond$ is an even function, we then see that the corrections to the propagator equation appear first time at the second order in gradients [70]. Therefore, to the first order in gradients, we have simply

$$\Delta^{r,a} = (\Omega_\phi^2 \pm i\Gamma_\phi)^{-1}, \tag{3.6}$$

or equivalently (*cf.* (2.10))

$$\Delta_h \;=\; \frac{\Omega_\phi^2}{\Omega_\phi^4 + \Gamma_\phi^2} \tag{3.7}$$

$$\mathcal{A}_\phi \;=\; \frac{\Gamma_\phi}{\Omega_\phi^4 + \Gamma_\phi^2}. \tag{3.8}$$

The kinetic and constraint equations can be obtained from the Kadanoff-Baym equations (3.2) as the hermitean and antihermitean parts,

$$-\diamond\big\{\Omega_\phi^2\big\}\big\{i\Delta^{<,>}\big\} + \diamond\big\{i\Pi^{<,>}\big\}\big\{\Delta_h\big\} \;=\; \frac{1}{2}\big(\Pi^>\Delta^< - \Pi^<\Delta^>\big). \tag{3.9}$$

$$\Omega_\phi^2\Delta^{<,>} + \frac{i}{2}\diamond\big(\big\{\Pi^>\big\}\big\{\Delta^<\big\} - \big\{\Pi^<\big\}\big\{\Delta^>\big\}\big) \;=\; \Pi^<\Delta_h, \tag{3.10}$$

where we truncated to second and first order in gradients, respectively. Because of the hermiticity property (2.50) of the scalar Wigner function, both kinetic and constraint equations must be satisfied simultaneously. The constraint equation selects the physical set of solutions among all solutions of the kinetic equation.

Note that the equations (3.7-3.8) are only formal solutions in the sense that Π_h and Γ_ϕ are functionals of $i\Delta^{<,>}$ and $iS^{<,>}$. Consequently, equations (3.9-3.10) are complicated integro-differential equations. However, in many cases deviations from equilibrium are small, so that one can linearize them in deviation from equilibrium, such that Π_h and Γ_ϕ can be computed by using the equilibrium distribution functions (2.64-2.73).

Inspired by the spectral form of the equilibrium solutions, we shall assume that close to equilibrium the following spectral decomposition holds

$$i\Delta^{<} = 2\mathcal{A}_\phi n_\phi, \qquad i\Delta^{>} = 2\mathcal{A}_\phi(1+n_\phi)\,, \tag{3.11}$$

where $n_\phi = n_\phi(k,x)$ is some unknown function representing a generalized distribution function on phase space. Similarly, for the self-energies, we can formally write

$$i\Pi^{<} \equiv 2\Gamma_\phi n_\Pi, \qquad i\Pi^{>} \equiv 2\Gamma_\phi(1+n_\Pi). \tag{3.12}$$

We have adopted a somewhat nonstandard definition for Γ_ϕ; the more standard definition is obtained by the simple replacement $\Gamma_\phi \to 2k_0\Gamma_\phi$. At the moment n_Π is just an arbitrary variable replacing $\Pi^{<,>}$, but it will become closely related to the particle distribution function in the end. It is important to note that unlike n_ϕ, it is *not* to be considered a free variable in the equations.
Then using the definitions (3.11-3.12) and the identity

$$\diamond\{f\}\{gh\} = g\diamond\{f\}\{h\} + h\diamond\{f\}\{g\}, \tag{3.13}$$

we rewrite Eqs. (3.9-3.10) as

$$\begin{aligned}
\mathcal{A}_\phi \diamond \{\Omega_\phi^2\}\{n_\phi\} + \Gamma_\phi \diamond \{\Delta_h\}\{n_\phi\} &= \Gamma_\phi\mathcal{A}_\phi\big(n_\phi - n_\Pi\big) &(3.14)\\
\Omega_\phi^2\mathcal{A}_\phi n_\phi + \diamond\big\{\Gamma_\phi\mathcal{A}_\phi\big\}\{n_\phi\} &= \Gamma_\phi n_\Pi\Delta_h. &(3.15)
\end{aligned}$$

We can now see that taking $n_\Pi \to n_\phi$ solves the kinetic equation (3.14) to the zeroth order in gradients, establishing the promised connection between n_Π and n_ϕ. This also shows that, to the first order in gradients, it is consistent to replace n_Π by the zeroth order solution n_ϕ, whenever the $\diamond$ operator acts on n_Π. In fact, we already took this into account by replacing n_Π by n_ϕ in the Γ_ϕ-terms on the *l.h.s.* of Eqs. (3.14-3.15).

We shall now see how to reduce these equations to an on-shell approximation, and in particular ask to what extent the on-shell limit is a good approximation to the plasma dynamics close to equilibrium. The equations can be simplified further when the width Γ_ϕ is small. While this is in general true in the weak coupling limit, due care must be exercised in making this approximation, because both Π_h and Γ_ϕ are often controlled by the same coupling constants.

3.1. Propagator equation

In the weak coupling limit $\Gamma_\phi \to 0$ and to first order in gradients, the spectral function (3.8) reduces to the singular spectral form,

$$\mathcal{A}_\phi \overset{\Gamma_\phi \to 0}{\longrightarrow} \mathcal{A}_s = \pi \,\mathrm{sign}(k_0)\, \delta\big(\Omega_\phi^2\big), \tag{3.16}$$

where the correction is of order Γ_ϕ. This singular hypersurface defines the usual quasiparticle dispersion relation

$$\Omega_\phi^2 \equiv k_0^2 - \vec{k}^2 - M^2 - \Pi_h(k_0, \vec{k}, x) = 0, \tag{3.17}$$

which defines the spectrum of the quasiparticle excitations of the system. This equation has in general two distinct solutions $k_0 = \omega_k$ and $k_0 = -\bar{\omega}_k$, corresponding to particles and antiparticles, respectively. Thus the spectral function (3.16) breaks up into two clearly separate contributions:

$$\mathcal{A}_s = \frac{\pi}{2\omega_k} Z_k\, \delta(k_0 - \omega_k) - \frac{\pi}{2\bar{\omega}_k} \bar{Z}_k\, \delta(k_0 + \bar{\omega}_k), \tag{3.18}$$

where the wave function renormalization factors Z_k and $\bar{Z}_k$ are

$$2\omega_k Z_k^{-1} = \big|\partial\Omega_\phi^2/\partial k_0\big|_{k_0=\omega_k}, \qquad 2\bar{\omega}_k \bar{Z}_k^{-1} = \big|\partial\Omega_\phi^2/\partial k_0\big|_{k_0=-\bar{\omega}_k}. \tag{3.19}$$

The definition of Δ_h in (3.7) becomes more problematic in the limit $\Gamma_\phi \to 0$ because of the appearance of the so called Landau ghost poles, when perturbative expressions are used for the self-energy Π_h [70]. Therefore, when needed, we shall relate Δ_h to $\mathcal{A}_\phi$ *via* equations (3.4-3.5) before taking the zero width limit. Let us note however that the spectral representation

$$\Delta_h(k,x) \overset{\Gamma \to 0}{\longrightarrow} \mathrm{Re} \int dk_0' \frac{\mathcal{A}_\phi(k_0', \vec{k}, x)}{k_0 - k_0' + i\epsilon} \tag{3.20}$$

remains valid for Δ_h even in the on-shell limit [70].

The full spectral function $\mathcal{A}_\phi$ must obey the sum-rule (2.59). The fact that the singular function $\mathcal{A}_s$ (3.18) fails this rule when interactions are included, tells us something important about the physical nature of the quasiparticle approximation. Indeed, one is here describing the spectrum of inherently collective plasma excitations by a simple single particle picture. The amount by which $\mathcal{A}_s$ fails the sum-rule can then be attributed to the effect of collective plasma excitations. This gives a quantitative estimate for the goodness, but also indicates the limits of applicability of the quasiparticle approximation.

Since both Π_h and Γ_ϕ are controlled by the same coupling constants, they typically acquire contributions at the same order in the coupling constant, so that one cannot take the limit $\Gamma_\phi \to 0$ as an independent approximation. The theories containing tadpoles, such as the $\lambda\phi^4$-theory, seem at a first sight to be an exception. However, the tadpoles give rise to a singular non-dissipative contribution to the self-energy, which can be absorbed by the mass term, implying that they cannot be responsible for thermalization. This means that in order to study thermalization properly, one has to include the higher order nonlocal dissipative contributions, which typically contribute to both the self-energy and width at the same order in the coupling constant. Consequently, there is a danger that one

might lose the consistency of the single particle picture. Nevertheless, as we shall now see, this is not the case. Namely, the two quantities appear in the spectral function in quite distinct ways such that the kinetic equation allows a consistent on-shell limit assuming equal accuracy in the computation of Π_h and Γ_ϕ.

3.2. Kinetic equation in the on-shell limit

Our aim is now to study the conditions under which the kinetic and constraint equations (3.14-3.15) reduce to the on-shell approximation for the effective plasma degrees of freedom found in the previous section. Bearing in mind the discussion concerning consistency of the quasiparticle limit, we start by using the expressions (3.7-3.8) with a finite width Γ_ϕ. The term linear in Γ_ϕ on the *l.h.s.* of (3.14) is often simply dropped without a careful consideration [71]. However, making use of the identity $\Gamma_\phi \Delta_h = \Omega_\phi^2 \mathcal{A}_\phi$, which is accurate to the leading order in gradients (*cf.* Eq. (3.15)), and expressions (3.7-3.8), we can in fact combine it with the first term in the kinetic equation to obtain

$$\frac{-2\Gamma_\phi^3}{(\Omega_\phi^4 + \Gamma_\phi^2)^2} \diamond \{\Omega_\phi^2\}\{n_\phi\} - \frac{2\Gamma_\phi^2 \Omega_\phi^2}{(\Omega_\phi^4 + \Gamma_\phi^2)^2} \diamond \{\Gamma_\phi\}\{n_\phi\} = \Gamma_\phi \mathcal{A}_\phi (n_\phi - n_\Pi) \,. \tag{3.21}$$

Note that in the weak coupling limit and to leading order in gradients

$$\frac{-2\Gamma_\phi^3}{(\Omega_\phi^4 + \Gamma_\phi^2)^2} \xrightarrow{\Gamma_\phi \to 0} \mathcal{A}_s, \tag{3.22}$$

such that Eq. (3.21) can be rewritten as

$$\mathcal{A}_s \diamond \{\Omega_\phi^2\}\{n_\phi\} - 2\Gamma_\phi \mathcal{A}_s \Delta_h \diamond \{\Gamma_\phi\}\{n_\phi\} = \Gamma_\phi \mathcal{A}_\phi (n_\phi - n_\Pi) \,. \tag{3.23}$$

Under the reasonable assumption that Γ_ϕ is a smooth function of the energy, we get from (3.23) a consistent expansion of the final result in powers of Γ_ϕ evaluated at the pole. In this expansion the contributions from the off-shellness cancel so that up to order $\mathcal{O}(\Gamma_\phi)$ the integrated equation involves only the leading on-shell excitations, defined by the singular spectral function (3.18). In other words, the integrated kinetic equation exhibits the important property of *closure* onto the on-shell excitations in the weak coupling limit. We can then replace (3.23) by the singular *on-shell kinetic equation*

$$\mathcal{A}_s \diamond \{\Omega_\phi^2\}\{n_\phi\} = \Gamma_\phi \mathcal{A}_s (n_\phi - n_\Pi). \tag{3.24}$$

We again emphasize that it is actually consistent to include non-local loop contributions to Π_h on the *l.h.s.* despite the fact that the singular single particle limit does not, strictly speaking, exist for the excitations. This is simply because the corrections due to off-shellness cancel up to $\mathcal{O}(\Gamma_\phi)$. This is fortunate, because applying the weak coupling limit in the strict sense to (3.24) indeed requires either computing Π_h and Γ_ϕ to the same order, or simply neglecting Γ_ϕ, leading to a collisionless Boltzmann equation. A somewhat orthogonal approach has been taken by Leupold [72] in which, based on particle number conservation, the author advocates a modification in the relation between the Wigner function and the on-shell distribution function.

The final step is to show that the constraint equation (3.15) can be written as

$$\Omega_\phi^2 \mathcal{A}_s (n_\phi - n_\Pi) = - \diamond \{\Gamma_\phi \mathcal{A}_s\}\{n_\phi\}. \tag{3.25}$$

In the limit $\Gamma_\phi \to 0$ the constraint equation becomes identically solved when the quasiparticle on-shell condition (3.17) is satisfied. In other words, the on-shell condition and constraint equations become degenerate. Note that we can here refer to the on-shell condition, not necessarily restricted to the singular approximation, but in the broader sense we derived the on-shell kinetic equation (3.24). Indeed, the constraint equation can always be satisfied by the on-shell solutions as given by Eq. (3.24), whereby it only gives rise to an integral constraint for the off-shell excitations. These however, do not belong to the set of dynamical degrees of freedom for the kinetic equation (3.24), which only contains the on-shell excitations as defined by the singular overall projection operator $\mathcal{A}_s$. In this sense the kinetic equation (3.24) and the constraint equation decouple. Of course, the off-shell excitations in the constraint equation (3.25) become relevant when the off-shell effects are included into the kinetic equation.

We emphasise that the derivation of the on shell kinetic equation (3.24) applies to the special case of one scalar field in weak coupling limit and close to equilibrium. It would be of interest to extend the analysis to the case of mixing scalar and fermionic fields.

4. KINETICS OF SCALARS: TREE-LEVEL ANALYSIS

In this section we analyze the tree-level dynamics of the scalar sector of our theory with mixing and emphasise its ramifications. The hermiticity property (2.50) of the scalar Wigner function implies that the hermitean and the antihermitean part of the equation of motion

$$\left(k^2 - \frac{1}{4}\partial^2 + ik\cdot\partial - M^2(x)e^{-\frac{i}{2}\overleftarrow{\partial}\cdot\partial_k}\right) i\Delta^< = 0 \tag{4.1}$$

ought to be satisfied simultaneously. A simple analysis reveals that the hermitean part corresponds to the *kinetic* equation, while the antihermitean part corresponds to the *constraint* equation. Roughly speaking, the kinetic equation describes the dynamics of quantum fields, while the constraint equation constrains the space of solutions of the kinetic equation [52]. In the simple case when $N = 1$ it is immediately clear that the first quantum correction to the constraint equation, the real part of (4.1), is of second order and to the kinetic equation (imaginary part) of third order in gradients or second order in $\hbar$. To extract the spectral information to second order in $\hbar$ is quite delicate since the constraint equation in (4.1) contains derivatives. The situation is more involved in the case of more than one mixing field. In this case it is convenient to rotate into the mass eigenbasis

$$M_d^2 = UM^2U^\dagger, \tag{4.2}$$

where U is the unitary matrix that diagonalizes M^2. In this propagating basis equation (4.1) becomes

$$\left(k^2 - \frac{1}{4}\mathcal{D}^2 + ik\cdot\mathcal{D} - M_d^2 e^{-\frac{i}{2}\overleftarrow{\mathcal{D}}\cdot\partial_k}\right)\Delta_d^< = 0, \tag{4.3}$$

where $\Delta_d^< \equiv U\Delta^<U^\dagger$ and the 'covariant' derivative is defined as

$$\mathcal{D}_\mu = \partial_\mu - i\,[\Xi_\mu,\ \cdot\ ]\,, \qquad \Xi_\mu = iU\partial_\mu U^\dagger\,. \tag{4.4}$$

Making use of $(i\Delta_d^<)^\dagger = i\Delta_d^<$, $\mathcal{D}_\mu^\dagger = \mathcal{D}_\mu$, which implies $(\mathcal{D}_\mu i\Delta_d^<)^\dagger = i\Delta_d^<\overleftarrow{\mathcal{D}}_\mu = \mathcal{D}_\mu i\Delta_d^<$, and $\hat{M}_{c,s}^2 i\Delta_d^< = \frac{1}{2}\{\hat{M}_{c,s}^2, i\Delta_d^<\} + \frac{1}{2}[\hat{M}_{c,s}^2, i\Delta_d^<]$, we can identify the antihermitean part of (4.3) as the constraint equation

$$\left(k^2 - \frac{1}{4}\mathcal{D}^2\right)i\Delta_d^< - \frac{1}{2}\left\{\hat{M}_c^2, i\Delta_d^<\right\} + \frac{i}{2}\left[\hat{M}_s^2, i\Delta_d^<\right] = 0, \tag{4.5}$$

and the hermitean part is the kinetic equation

$$k\cdot\mathcal{D}i\Delta_d^< + \frac{1}{2}\left\{\hat{M}_s^2, i\Delta_d^<\right\} + \frac{i}{2}\left[\hat{M}_c^2, i\Delta_d^<\right] = 0\,. \tag{4.6}$$

We defined

$$\begin{aligned}\hat{M}_c^2 &= M_d^2 \cos\frac{1}{2}\overleftarrow{\mathcal{D}}\cdot\partial_k\\ \hat{M}_s^2 &= M_d^2 \sin\frac{1}{2}\overleftarrow{\mathcal{D}}\cdot\partial_k\,,\end{aligned} \tag{4.7}$$

and $[\,\cdot\,,\cdot]$, $\{\,\cdot\,,\,\cdot\,\}$ denote a commutator and an anticommutator, respectively. The constraint (4.5) and kinetic equation (4.6) are formally still exact. Since we are interested in order $\hbar$ correction to the classical approximation, it is good to keep in mind that one can always restore $\hbar$ dependencies

by the simple replacements $\partial \to \hbar\partial$ and $i\Delta_d^< \to \hbar^{-1} i\Delta_d^<$. We work to order $\hbar$ accuracy, so it suffices to truncate the constraint equation (4.5) to first order in gradients, and the kinetic equation (4.6) to second order in gradients:

$$k^2 i\Delta_d^< - \frac{1}{2}\left\{M_d^2, i\Delta_d^<\right\} + \frac{i}{4}\left[\mathcal{D}M_d^2, \partial_k i\Delta_d^<\right] \;=\; 0, \tag{4.8}$$

$$k\cdot\mathcal{D}\, i\Delta_d^< + \frac{1}{4}\left\{\mathcal{D}M_d^2, \partial_k i\Delta_d^<\right\} + \frac{i}{2}\left[M_d^2\Big(1 - \frac{1}{8}(\overleftarrow{\mathcal{D}}\cdot\partial_k)^2\Big), i\Delta_d^<\right] \;=\; 0\,. \tag{4.9}$$

We shall now consider these equations in more detail. The off-diagonal elements of $i\Delta_d^<$ in both the constraint (4.8) and the kinetic equation (4.9) are sourced by the diagonal elements through the terms involving commutators, which are suppressed by at least $\hbar$ with respect to the diagonal elements [80]. On the other hand, the off-diagonals source the diagonal equations through terms that are of the same order as the diagonals. For the CP-violating diagonal densities that are suppressed at least by one power of $\hbar$, this immediately implies that the CP-violating off-diagonals are at least of the order $\hbar^2$, and thus cannot source the diagonal densities at order $\hbar$. By a similar argument, the second order term in the commutator in (4.9) can be dropped, since it can induce effects only at order $\hbar^2$.

This analysis is however incomplete for the following reason. When the rotation matrices $\Xi_\mu = iU\partial_\mu U^\dagger$ in (4.4) contain CP-violation, the off-diagonal elements can in principle be CP-violating already at order $\hbar$, and hence source the diagonals at the same order. In order to get more insight into the role of the off-diagonals, we now analyze the case of two mixing scalars. For notational simplicity we omit the index d for diagonal in the following. All quantities have to be taken in the rotated basis, where the mass is diagonal. The constraint equations (4.8), when written in components, are

$$\Big(k^2 - M_{\mathtt{ii}}^2\Big) i\Delta_{\mathtt{ii}}^< \mp \frac{1}{4}\delta(M^2)\Big(\Xi_{\mathtt{12}}\cdot\partial_k i\Delta_{\mathtt{21}}^< + \Xi_{\mathtt{21}}\cdot\partial_k i\Delta_{\mathtt{12}}^<\Big) \;=\; 0, \tag{4.10}$$

$$\Big(k^2 - \bar{M}^2 + \frac{i}{4}(\partial\delta(M^2))\cdot\partial_k\Big) i\Delta_{\mathtt{12}}^< + \frac{1}{4}\delta(M^2)\Xi_{\mathtt{12}}\cdot\partial_k\delta(i\Delta^<) \;=\; 0\,, \tag{4.11}$$

where $\bar{M}^2 \equiv \mathrm{Tr}(M^2)/2 = (M_{\mathtt{11}}^2 + M_{\mathtt{22}}^2)/2$, $\delta(M^2) \equiv M_{\mathtt{11}}^2 - M_{\mathtt{22}}^2$, $\delta(i\Delta^<) \equiv i\Delta_{\mathtt{11}}^< - i\Delta_{\mathtt{22}}^<$, and the equation for $i\Delta_{\mathtt{21}}^< = (i\Delta_{\mathtt{12}}^<)^*$ is obtained by taking the complex conjugate of (4.11). To order $\hbar$ the diagonal equations (4.10) are solved by the spectral on-shell solution

$$\begin{aligned} i\Delta_{\mathtt{ii}}^<(k,x) \;&=\; 2\pi\delta\Big(k^2 - M_{\mathtt{ii}}^2\Big)\mathrm{sign}(k_0) n_i^\phi(k,x) \\ &=\; \frac{\pi}{\omega_{\phi i}}\left[\delta(k_0 - \omega_{\phi i}) - \delta(k_0 + \omega_{\phi i})\right] n_i^\phi(k,x)\,, \end{aligned} \tag{4.12}$$

where $\omega_{\phi i} = [\vec{k}^{\,2} + M_{\mathtt{ii}}^2]^{1/2}$, and $n_i^\phi(k,x)$ represents the occupation number density on phase space, which in thermal equilibrium reduces to $n_i^\phi(k,x) \to n_{\rm eq}^\phi = 1/(\mathrm{e}^{\beta k_0} - 1)$. By making use of the sum rule (2.59) for the spectral function $\mathcal{A}_\phi = (i/2)(\Delta^> - \Delta^<)$, one can show that the other Wigner function has to be of the form

$$i\Delta_{\mathtt{ii}}^>(k,x) = 2\pi\delta\Big(k^2 - M_{\mathtt{ii}}^2\Big)\mathrm{sign}(k_0)\Big(1 + n_i^\phi(k,x)\Big) \tag{4.13}$$

with the same density n_i^ϕ as in (4.12). These solutions are consistent provided the off-diagonals $i\Delta_{\mathtt{12}}^<$ and $i\Delta_{\mathtt{21}}^<$ are of order $\hbar$, which, as we shall argue, is a self-consistent assumption. The off-diagonal

constraint equation (4.11) is solved by

$$i\Delta^<_{12} = n^\phi_{12}\,\delta(k^2-\bar{M}^2) - \frac{1}{2}\Xi_{12}\cdot\partial_k \mathrm{Tr}(i\Delta^<) - \frac{2}{\delta(M^2)}\,k\cdot\Xi_{12}\,\delta(i\Delta^<)\,, \tag{4.14}$$

where $n^\phi_{12} = n^\phi_{12}(k,x)$ is a function that can be determined from the boundary conditions. Since we are interested in situations close to equilibrium in which the off-diagonals are driven away from zero primarily by the diagonals, we can set n^ϕ_{12} to zero for our purposes. Note that the solution (4.14) contains derivatives of the delta function, and hence it does not strictly speaking represent an on-shell form.

Consider now the kinetic equations (4.9), which in components read

$$\begin{aligned}\Big(k\cdot\partial + \frac{1}{2}(\partial M^2_{\mathtt{ii}})\cdot\partial_k\Big)i\Delta^<_{\mathtt{ii}} \;&=\; \pm i\,k\cdot\Big(\Xi_{12}i\Delta^<_{21} - \Xi_{21}i\Delta^<_{12}\Big)\\ &-\;\frac{i}{4}\delta(M^2)\Big(\Xi_{12}\cdot\partial_k i\Delta^<_{21} - \Xi_{21}\cdot\partial_k i\Delta^<_{12}\Big)\end{aligned} \tag{4.15}$$

$$\Big(k\cdot\partial + \frac{1}{2}(\partial\bar{M}^2)\cdot\partial_k + \frac{i}{2}\delta(M^2) - ik\cdot\delta(\Xi)\Big)i\Delta^<_{12} = -ik\cdot\Xi_{12}\delta(i\Delta^<) - \frac{i}{4}\delta(M^2)\Xi_{12}\cdot\partial_k\mathrm{Tr}(i\Delta^<)\,, \tag{4.16}$$

and the equation for $i\Delta^<_{21}$ is again obtained by taking the complex conjugate of (4.16). To leading order in gradients the off-diagonal equation (4.16) is solved by

$$i\Delta^<_{12} = -\frac{1}{2}\,\Xi_{12}\cdot\partial_k\mathrm{Tr}(i\Delta^<) - \frac{2}{\delta(M^2)}\,k\cdot\Xi_{12}\,\delta(i\Delta^<)\,, \tag{4.17}$$

and similarly for $i\Delta^<_{21}$. Remarkably, this corresponds precisely to the constraint equation solution (4.14), provided one sets $n^\phi_{12}=0$, representing a very nontrivial consistency check of the kinetic theory for mixing particles. By making use of the technique of Green functions, in Appendix A we show that the leading order result (4.17) represents a valid approximation, provided the condition for the gradient approximation $k\cdot\partial \ll \delta(M^2)$ is satisfied, that is one is not near the degenerate mass limit, $\delta(M^2)=0$.

The example of mixing scalars represents indeed a nice illustration of the workings of the constraint and kinetic equations. Unlike in the one field case, in which the kinetic flow term can be obtained by acting with the bilinear $\diamond$ operator on the constraint equation, in the mixing case the situation is more complex. All of the solutions of the kinetic equation (4.16) are simultaneously solutions of the constraint equation (4.11). The converse is however *not* true: the constraint equation contains a larger set of solutions, which include the homogeneous solutions (4.14) that lie on a different energy shell, which are in fact excluded by imposing the kinetic equation (4.16). This can be shown as follows: the constraint equation (4.11) can be obtained from the kinetic equation (4.16) by multiplying it by $\big(k^2-\mathrm{Tr}(M^2)/2\big)/\delta(M^2)$, followed by a partial integration. The constraint equation obtained this way has of course a larger set of solutions. The additional solutions are of the type: a function multiplying $\delta\big(k^2-\mathrm{Tr}(M^2)/2\big)$.

Finally, upon inserting (4.17) into the diagonal equation (4.15), we get

$$\Big(k\cdot\partial + \frac{1}{2}(\partial M^2_{\mathtt{ii}})\cdot\partial_k \pm i\,k\cdot\big[\Xi_{12},\Xi_{21}\big]\cdot\partial_k\Big)i\Delta^<_{\mathtt{ii}} = 0\,. \tag{4.18}$$

It is remarkable that not only the off-diagonals have disappeared from (4.18), but also the diagonals decouple. The commutator $k \cdot \Big[\Xi_{12}, \Xi_{21}\Big] \cdot \partial_k$ does *not* vanish in general. However, for planar walls and in the wall frame,

$$k \cdot \Big[\Xi_{12}, \Xi_{21}\Big] \cdot \partial_k \;\rightarrow\; -k_z\Big(\Xi_{z12}\,\Xi_{z21} - \Xi_{z21}\,\Xi_{z12}\Big)\partial_{k_z} = 0 \tag{4.19}$$

vanishes identically, so that Eq. (4.18) reduces to

$$\Big(k \cdot \partial - \frac{1}{2}(\partial_z M^2_{\mathtt{ii}})\partial_{k_z}\Big) i\Delta^<_{\mathtt{ii}} = 0\,. \tag{4.20}$$

This proves that, although a naive analysis of Eqs. (4.8-4.9) indicated that there might be a CP-violating source in the kinetic equation at order $\hbar$, a more detailed look at the structure of these equations shows that for planar walls considered in the wall frame *no* such source appears. Equation (4.20) thus represents a self-consistent kinetic description of scalar fields accurate to order $\hbar$, such that the only source in the kinetic equation is the classical force, $\vec{F}_i = -\nabla\omega_{\phi i} = -\nabla M^2_{\mathtt{ii}}/2\omega_{\phi i}$. In addition, we have shown that to order $\hbar$ the on-shell approximation for the diagonal occupation numbers (4.12) with the classical dispersion relation $k_0 = \pm\omega_{\phi i}$ still holds. Even though the on-shell approximation fails for the off-diagonals, this is only relevant for the dynamics at higher orders in gradient expansion. Further, inclusion of collisions in the off-diagonal equations cannot change our conclusions concerning the source cancellation expressed in Eq. (4.18), even though it may introduce new collisional contributions. (For a detailed study of collisional sources we refer to Paper II.) Needless to say, this analysis easily generalizes to the case of N mixing scalars. This completes the proof that was originally presented in Ref. [52], which states that in the flow term of scalars there is no source for baryogenesis at order $\hbar$ in gradient expansion.

4.1. Boltzmann transport equation for CP-violating scalar densities

For completeness, and to make a connection between the tree-level analysis presented here and the analysis of the collision term in Paper II, we now show how – starting with the kinetic equation for the Wightman function (4.20) – one obtains a Boltzmann equation for the CP-violating scalar particle densities. All flavor matrices are to be taken in the basis where the mass is diagonal.
In the beginning we review some of the basics of the C (charge) and CP (charge and parity) transformations of quantum scalar fields. Under C and P, a scalar field ϕ transforms as

$$\begin{aligned}\phi^c(u) &\equiv \mathcal{C}\,\phi(u)\,\mathcal{C}^{-1} = \xi^*_\phi\phi^*(u)\;\;,\;\;\mathcal{C}\,\phi^\dagger(u)\,\mathcal{C}^{-1} \;= \xi_\phi\phi^T(u)\\ \phi^p(u) &\equiv \mathcal{P}\,\phi(u)\,\mathcal{P}^{-1} = \eta^*_\phi\phi(\bar{u})\;\;,\;\;\mathcal{P}\,\phi^\dagger(u)\,\mathcal{P}^{-1} = \eta_\phi\phi^\dagger(\bar{u})\,,\end{aligned} \tag{4.21}$$

where $|\xi_\phi| = 1$ and $|\eta_\phi| = 1$ are global phases, and $\bar{u}^\mu = (u_0, -\vec{u})$ denotes the inversion of the spatial part of u. Note that our definition of charge conjugation includes an additional transposition with respect to the usual definition [73], which is required in the case of mixing scalar fields, when ϕ is a vector in flavor space. From (4.21) it follows that the scalar Wightman functions transform as

$$i\Delta^<(u,v) \;\xrightarrow{\;\mathcal{C}\;}\; i\Delta^>(v,u)^T \tag{4.22}$$

$$i\Delta^<(u,v) \;\xrightarrow{\;\mathcal{CP}\;}\; i\Delta^>(\bar{v},\bar{u})^T\,. \tag{4.23}$$

When written in the Wigner representation, the equivalent transformations are

$$i\Delta^{<}(k,x) \ \xrightarrow{\ \mathcal{C}\ }\ i\Delta^{>}(-k,x)^{T} \equiv i\Delta^{c<}(k,x) \tag{4.24}$$
$$i\Delta^{<}(k,x) \ \xrightarrow{\ \mathcal{CP}\ }\ i\Delta^{>}(-\bar{k},\bar{x})^{T} \equiv i\Delta^{cp<}(\bar{k},\bar{x})\,. \tag{4.25}$$

The CP-conjugate of the Wigner function $i\Delta^{<}(k,x)$ is related to physical objects at position $-\vec{x}$ with momentum $-\vec{k}$. Therefore we introduced additional inversions of the spatial parts of the position and momentum arguments in the definition of $i\Delta^{cp<}(k,x)$, so that this object indeed describes particles at position $\vec{x}$ and with momentum $\vec{k}$. Note that this definition differs from what is usually found in textbooks. In order to study the effects of the C and CP transformations on the distribution functions, we proceed analogously to (4.12) and define the spectral solutions

$$i\Delta^{c/cp<}_{\mathtt{ii}}(k,x) \ = \ \frac{\pi}{\omega_{\phi i}}\left[\delta(k_0-\omega_{\phi i})-\delta(k_0+\omega_{\phi i})\right] n_i^{\phi c/cp}(k,x) \tag{4.26}$$
$$i\Delta^{c/cp>}_{\mathtt{ii}}(k,x) \ = \ \frac{\pi}{\omega_{\phi i}}\left[\delta(k_0-\omega_{\phi i})-\delta(k_0+\omega_{\phi i})\right] \Big(1+n_i^{\phi c/cp}(k,x)\Big)\,. \tag{4.27}$$

Now upon inserting this and (4.12-4.13) into (4.24) and (4.25), we find

$$n_i^{\phi c}\big(\omega_{\phi i},\vec{k},x\big) = n_i^{\phi cp}\big(\omega_{\phi i},\vec{k},x\big) = -\big[1+n_i^{\phi}\big(-\omega_{\phi i},-\vec{k},x\big)\big]\,. \tag{4.28}$$

Because of the additional inversions in the definition of $i\Delta^{cp<}$, the two densities are identical.

We shall now consider the kinetic equation for scalars. First we include the collision term into Eq. (4.20),

$$\Big(k\cdot\partial-\frac{1}{2}(\partial_z M^2_{\mathtt{ii}})\partial_{k_z}\Big)i\Delta^{<}_{\mathtt{ii}}(k,x) = \frac{1}{2}\big[\mathcal{C}_{\phi\mathtt{ii}}(k,x)+\mathcal{C}^{\dagger}_{\phi\mathtt{ii}}(k,x)\big]\,, \tag{4.29}$$

where $\mathcal{C}_{\phi\mathtt{ii}}$ denotes the diagonal elements of the collision term defined in Eq. (2.56) after rotation into the basis where the mass is diagonal. Next, we insert the spectral on-shell solution (4.12) and integrate over positive frequencies, to obtain

$$\left(\partial_t+\frac{\vec{k}\cdot\nabla}{\omega_{\phi i}}-\frac{(\partial_z M^2_{\mathtt{ii}}(z))}{2\omega_{\phi i}}\partial_{k_z}\right) f^{\phi}_{i+}(\vec{k},x) = \frac{1}{2\pi}\int_0^{\infty} dk_0\,\big[\mathcal{C}_{\phi\mathtt{ii}}(k,x)+\mathcal{C}^{\dagger}_{\phi\mathtt{ii}}(k,x)\big]\,, \tag{4.30}$$

which is the kinetic equation for the scalar distribution function

$$f^{\phi}_{i+}(\vec{k},x) \equiv n_i^{\phi}\big(\omega_{\phi i}(\vec{k},x),\vec{k},x\big)\,. \tag{4.31}$$

Since $i\Delta^{>}(k,x)$ and $i\Delta^{<}(k,x)$ satisfy identical equations, we can immediately write

$$\Big(-k\cdot\partial+\frac{1}{2}(\partial_z M^2_{\mathtt{ii}}(z))\partial_{k_z}\Big)i\Delta^{cp<}_{\mathtt{ii}}(k,x) = \frac{1}{2}\big[\mathcal{C}^{*}_{\phi\mathtt{ii}}(-k,x)+\mathcal{C}^{T}_{\phi\mathtt{ii}}(-k,x)\big]\,, \tag{4.32}$$

where we took account of (4.25). We can omit the transposition in the collision term, because we are only interested in the diagonal elements. Upon integrating over positive frequencies this becomes

$$-\left(\partial_t+\frac{\vec{k}\cdot\nabla}{\omega_{\phi i}}-\frac{(\partial_z M^2_{\mathtt{ii}}(z))}{2\omega_{\phi i}}\partial_{k_z}\right) f^{\phi}_{i-}(\vec{k},x) = \frac{1}{2\pi}\int_0^{\infty} dk_0\,\big[\mathcal{C}_{\phi\mathtt{ii}}(-k,x)+\mathcal{C}^{\dagger}_{\phi\mathtt{ii}}(-k,x)\big]\,. \tag{4.33}$$

This transport equation is the CP-conjugate of (4.30) for antiparticles, where we defined the antiparticle distribution function as

$$f_{i-}^{\phi}(\vec{k},x) \equiv n_i^{\phi cp}\big(\omega_{\phi i},\vec{k},x\big) = -\big[1+n_i^{\phi}\big(-\omega_{\phi i},-\vec{k},x\big)\big]\,. \tag{4.34}$$

This then implies the following definition for the CP-violating particle density:

$$\delta f_i^{\phi} = f_{i+}^{\phi} - f_{i-}^{\phi}\,. \tag{4.35}$$

The relevant kinetic equation for δf_i^{ϕ} is obtained simply by summing (4.30) and (4.33),

$$\begin{aligned}\left(\partial_t + \frac{\vec{k}\cdot\nabla}{\omega_{\phi i}} - \frac{(\partial_z M_{\tt ii}^2(z))}{2\omega_{\phi i}}\partial_{k_z}\right)\delta f_i^{\phi}(\vec{k},x) \;=\; & \frac{1}{2\pi}\int_0^{\infty} dk_0\,\big[\mathcal{C}_{\phi\tt ii}(k,x)+\mathcal{C}^{\dagger}_{\phi\tt ii}(k,x) \\ & +\mathcal{C}_{\phi\tt ii}(-k,x)+\mathcal{C}^{\dagger}_{\phi\tt ii}(-k,x)\big]\,. \end{aligned}\tag{4.36}$$

It should be stressed that the absence of a CP-violating force in the flow term at linear order in $\hbar$ is made explicit in this equation. Further, this equation makes it transparent how to extract CP-violating contributions from the collision term.

4.2. Applications to the stop sector of the MSSM

This analysis is relevant for example for a calculation of the CP-violating force in the *stop* sector $\tilde{q}=(\tilde{t}_L,\tilde{t}_R)^T$ of the MSSM [36, 39, 46, 47, 48, 49], in which the mass matrix reads

$$M_{\tilde{q}}^2 = \begin{pmatrix} m_Q^2 & y(A^*H_2+\mu H_1) \\ y(AH_2+\mu^* H_1) & m_U^2 \end{pmatrix}, \tag{4.37}$$

where m_Q^2 and m_U^2 denote the sum of the soft SUSY-breaking masses, including D-terms and $m_t^2 = y^2H_2^2$. Our analysis immediately implies that, for squarks in the quasiparticle picture, there is no CP-violating correction to the dispersion relation at first order in gradients, and hence there is no CP-violating semiclassical force in the flow term of the kinetic equation at order $\hbar$. This is in contrast to what was found in Refs. [46, 47, 48, 49].

To investigate whether there is CP-violation in the off-diagonal sector of the theory, we note that the mass matrix is diagonalized by the unitary matrix

$$U = \begin{pmatrix} \cos\theta & -\sin\theta\,\mathrm{e}^{-i\sigma} \\ \sin\theta\,\mathrm{e}^{i\sigma} & \cos\theta \end{pmatrix}, \tag{4.38}$$

where

$$\tan 2\theta = \frac{2y|A^*H_2+\mu H_1|}{m_Q^2-m_U^2}, \quad \tan\sigma = \frac{|A|\tan\beta\sin\alpha + |\mu|\sin\zeta}{|A|\tan\beta\cos\alpha+|\mu|\cos\zeta}, \tag{4.39}$$

with $A^*H_2+\mu H_1 = |A^*H_2+\mu H_1|\mathrm{e}^{i\sigma}$, $A^*=|A|\mathrm{e}^{i\alpha}$, $\mu=|\mu|\mathrm{e}^{i\zeta}$ and $\tan\beta = H_2/H_1$. From this we easily obtain

$$\Xi_\mu \equiv iU\partial_\mu U^\dagger = \begin{pmatrix} 0 & i\mathrm{e}^{-i\sigma} \\ -i\mathrm{e}^{i\sigma} & 0 \end{pmatrix}\partial_\mu\theta + \frac{1}{2}\begin{pmatrix} -(1-\cos 2\theta) & \sin 2\theta\,\mathrm{e}^{-i\sigma} \\ \sin 2\theta\,\mathrm{e}^{i\sigma} & 1-\cos 2\theta \end{pmatrix}\partial_\mu\sigma. \tag{4.40}$$

Since the off-diagonal elements of Ξ_μ are precisely the ones involved in the mixing of the diagonal and off-diagonal equations, Ξ_μ is CP-violating whenever $\sigma \neq 0$ and $\tan\beta$ varies at the phase interface, which is in general realized at the phase transition in the MSSM by complex A and μ parameters.

5. KINETICS OF FERMIONS: TREE-LEVEL ANALYSIS

In this section we consider the dynamics of fermions in the presence of scalar and pseudoscalar space-time dependent mass terms. Our analysis is of relevance, for example, for electroweak baryogenesis calculations, and establishes the nature of the propagating quasiparticle states in the electroweak plasma at the first order electroweak phase transition. In supersymmetric models baryogenesis is typically mediated by charginos and neutralinos [46]. Since they both mix at the tree level, it is very important to consider the more general case of N mixing fermions [52]. We derive the dispersion relation and kinetic equation for mixing fermions accurate to order $\hbar$. We also construct the equilibrium Wigner function to order $\hbar$, which can then be used for baryogenesis source calculations. This section is not just an overview of our original work [52, 53], but it also contains new insights and results.

5.1. Spin conservation and the fermionic Wigner function

In section 2.4 we have already derived the equation of motion (2.55) for the fermionic Wigner function. For the moment we neglect the interactions induced by $\mathcal{L}_{\rm int}$ in the Lagrangean (2.13). The equations for $S^<$ and $S^>$ are identical, so we omit the label in the following, indicating that the equations hold for both of them. The equation of motion is

$$\mathcal{D}S \equiv \left(\not{k} + \frac{i}{2}\not{\partial} - (m_h + i\gamma^5 m_a)\mathrm{e}^{-\frac{i}{2}\overleftarrow{\partial}\cdot\partial_k}\right) S = 0, \tag{5.1}$$

where we introduced the symbol $\mathcal{D}$ for the kinetic derivative operator. For planar walls propagating in z-direction the mass terms in (5.1) simplify to $m_{h,a}(z)\mathrm{e}^{\frac{i}{2}\overleftarrow{\partial_z}\partial_{k_z}}$, when written in the wall frame. Equation (5.1) is the tree-level fermionic master equation, which contains all the necessary information to study the nature of the propagating fermionic states in the presence of a space-time dependent mass term. We shall now show that, for planar walls, Eq. (5.1) contains a conserved quantity, corresponding to the spin pointing in the z-direction in the rest frame of the particle. This will then allow us to recast $S^{<,>}$ in a form which is block diagonal in spin, such that different spin blocks completely decouple, provided, of course, they decouple at the boundaries. A very nontrivial consequence of this observation is that a *quasiparticle picture*, when formulated in terms of the spin states, survives to order $\hbar$ in gradient expansion.

5.1.1. *The 1+1 dimensional ($\vec{k}_\parallel = 0$) frame*

In order to establish what precisely is the conserved quantity, we consider a particle with arbitrary momentum $\vec{k}$ and perform a boost into the frame where the particle momentum parallel to the wall vanishes, while the momentum component perpendicular to the wall remains unaffected:

$$k = (k_0, k_x, k_y, k_z) \to \tilde{k} = \left(\tilde{k}_0, 0, 0, k_z\right) \quad , \quad \tilde{k}_0 = \mathrm{sign}(k_0)\sqrt{k_0^2 - \vec{k}_\parallel^2}\,. \tag{5.2}$$

In this frame the particle moves only perpendicular to the wall, which is why we call it "1+1 frame". The necessary boost is characterized by

$$\vec{v}_{\parallel} = \frac{\vec{k}_{\parallel}}{k_0}, \qquad \gamma_{\parallel} = \frac{k_0}{\tilde{k}_0}. \tag{5.3}$$

In the spinor space it is represented by the operator

$$L(k) = \frac{k_0 + \tilde{k}_0 - \gamma^0 \vec{\gamma} \cdot \vec{k}_{\parallel}}{[2\tilde{k}_0(k_0 + \tilde{k}_0)]^{1/2}} \tag{5.4}$$

and its inverse $L^{-1}(k_0, \vec{k}) = L(k_0, -\vec{k})$, which act like

$$L(k)\, \not{k}\, L^{-1}(k) \;=\; \gamma^0 \tilde{k}_0 - \gamma^3 k_z \;\equiv\; \tilde{\not{k}}\,. \tag{5.5}$$

The mass terms in the equation of motion are transformed as follows:

$$L(k) m_{h,a}(x) \mathrm{e}^{-\frac{i}{2}\overleftarrow{\partial}\cdot\partial_k} L^{-1}(k) = m_{h,a}(\tilde{x}) \exp\Big(-\frac{i}{2}\, \overleftarrow{\tilde{\partial}} \cdot \big(\partial_{\tilde{k}} + \big[L(k)\partial_k L^{-1}(k), \cdot\,\big]\big)\Big). \tag{5.6}$$

Since $L(k)$ is a function of k_0 and $\vec{k}_{\parallel}$, the commutator term does in general not even vanish for planar walls in the frames where $m = m(t,z)$. The exception is the wall frame of a planar wall, where $m = m(z)$, such that $(\partial m) \cdot (L(k)\partial_k L^{-1}(k)) = 0$, and the commutator in (5.6) vanishes. For this reason we work from now on in the *wall frame* of a planar wall. In section 5.2.4 below we remark on how to transform our results to other frames, in particular to the plasma frame. Upon transforming (5.1), we then get

$$\tilde{\mathcal{D}}\tilde{S} \equiv \left(\tilde{\not{k}} + \frac{i}{2}\tilde{\not{\partial}} - (m_h + i\gamma^5 m_a)\mathrm{e}^{\frac{i}{2}\overleftarrow{\partial_z}\partial_{k_z}} \right) \tilde{S} = 0 \quad (\texttt{1 + 1 dim. frame}), \tag{5.7}$$

where the boosted Wigner function is

$$\tilde{S}(\tilde{k}) = L(k) S(k) L^{-1}(k)\,. \tag{5.8}$$

Since the bubble wall, which is responsible for any dependence of the Wigner function on the average coordinate x^μ, is symmetric in the x-y plane, we can assume that the Wigner function has the same symmetry:

$$\tilde{S} = \tilde{S}(\tilde{k}, \tilde{t}, z)\,. \tag{5.9}$$

This form holds true for homogeneous boundary conditions, which is the case for most of the time during a first order phase transition, during which the bubbles are large, almost planar and far away from each other, when measured in the units of a typical diffusion scale. The derivative in (5.7) then reduces like $\tilde{\not{\partial}} \to \gamma^0 \partial_{\tilde{t}} + \gamma^3 \partial_z$, so that the spin operator in z-direction

$$\tilde{S}_z = \gamma^0 \gamma^3 \gamma^5 \tag{5.10}$$

commutes with the kinetic operator

$$[\tilde{\mathcal{D}}, \tilde{S}_z] = 0, \tag{5.11}$$

and hence in this frame the spin in z-direction is a good quantum number. Assuming that the plasma is in thermal equilibrium before the phase transition takes place, the fermions are described by the equilibrium Wigner function (2.71), which is diagonal in spin in the 1+1 frame. The interaction with the bubble wall introduces no spin mixing, so that we can write $\tilde{S}$ in the spin-block diagonal form

$$\tilde{S} = \sum_{s=\pm 1} \tilde{S}_s, \qquad \tilde{S}_s \equiv \tilde{P}_s \tilde{S} \tilde{P}_s, \tag{5.12}$$

where $\tilde{P}_s$ denotes the spin projector:

$$\tilde{P}_s = \frac{1}{2}(\mathbb{1} + s\tilde{S}_z), \qquad \tilde{P}_s \tilde{P}_{s'} = \delta_{ss'} \tilde{P}_s, \qquad s = \pm 1. \tag{5.13}$$

The spin diagonal Wigner function lives in the subalgebra of the the Clifford algebra of the Dirac matrices which is spanned by the matrices commuting with $\tilde{P}_s$. We can write down a decomposition of $\tilde{S}_s$ by using a suitable hermitean basis of the subalgebra,

$$\tilde{S}_s(\tilde{k}) = i\tilde{P}_s(k) \left[s\gamma^3\gamma^5 \tilde{g}^s_0(\tilde{k}) - s\gamma^3 \tilde{g}^s_3(\tilde{k}) + \mathbb{1}\tilde{g}^s_1(\tilde{k}) - i\gamma^5 \tilde{g}^s_2(\tilde{k}) \right], \tag{5.14}$$

with $\tilde{g}^s_a$, $a = 0, 1, 2, 3$ being scalar functions. If we choose the following chiral representation of the Dirac matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{5.15}$$

where $\sigma^\mu = (\mathbb{1}, \sigma^i)$, $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$, and σ^i, $i = 1, 2, 3$ are the usual 2×2 Pauli matrices, then the spin operator is

$$\tilde{S}_z = \gamma^0\gamma^3\gamma^5 = \mathbb{1} \otimes \sigma^3 \tag{5.16}$$

and the spin block diagonality of the Wigner function becomes explicit:

$$-i\gamma^0 \tilde{S}_s = \frac{1}{2}\rho^a \otimes (\mathbb{1} + s\sigma^3)\tilde{g}^s_a\,. \tag{5.17}$$

5.1.2. The 3+1 dimensional (moving) frame

Knowing that in the 1+1 frame the spin in z-direction is conserved by the interaction with the bubble wall, we can construct the corresponding conserved quantity in the original frame by boosting the spin operator $\tilde{S}_z$ back to the original frame:

$$S_z(k) \equiv L^{-1}(k)\tilde{S}_z L(k) = \frac{1}{\tilde{k}_0}\left(k_0\gamma^0 - \vec{k}_\parallel \cdot \vec{\gamma}\right)\gamma^3\gamma^5\,. \tag{5.18}$$

In order to clarify the physical meaning of this operator, we recall that the covariant form of the spin operator is given by the Pauli-Lubanski tensor

$$S_{PL}(k, n) \equiv -\frac{1}{e_0}\, \not{k}\, \not{n}\gamma^5\,, \qquad e_0 \equiv (k^2)^{1/2}\,, \tag{5.19}$$

which measures the spin of a particle with momentum k in the direction $\vec{n}$ ($n^2 = -1$, $n \cdot k = 0$). For simplicity we choose the normalization for S_{PL} such that the eigenvalues are ± 1. In literature one

often finds the normalization which corresponds to the eigenvalues $\pm\hbar/2$. In the on-shell limit, e_0 reduces to the particle's mass. In the rest frame the Pauli-Lubanski tensor becomes $S_{PL} \xrightarrow{\vec{k}\to 0} \gamma^0 \not{n}' \gamma^5$, where n' is the spin vector in the rest frame, which is related to n by the corresponding Lorentz boost. To measure the spin in z-direction in the 1+1-dimensional frame, we use

$$\tilde{k}^\mu = (\tilde{k}_0, 0, 0, k_z), \qquad \tilde{n}^\mu = \frac{1}{e_0}(k_z, 0, 0, \tilde{k}_0), \tag{5.20}$$

and we find, of course, $S_{PL}(\tilde{k}, \tilde{n}) = \gamma^0\gamma^3\gamma^5 = \tilde{S}_z$. The same operator measures also the spin in z-direction in the rest frame, because the spin operator (5.19) is invariant under boosts as long as the direction of the spin vector is parallel to the momentum, $\vec{n}\,||\vec{k}$. Now, by setting $S_{PL}(k,n)$ equal to $S_z(k)$ in (5.18), we find that S_z measures spin in the direction corresponding to

$$n^\mu(k) = \frac{1}{\tilde{k}_0 e_0} \begin{pmatrix} k_0 k_z \\ k_x k_z \\ k_y k_z \\ \tilde{k}_0^2 \end{pmatrix} . \tag{5.21}$$

The same result is of course obtained by boosting $\tilde{n}^\mu$ in (5.20) to the original (moving) frame k^μ. In the highly relativistic limit we have $\tilde{k}_0^2 \to k_z^2$, $k_0^2 \to \vec{k}^2$, and our special spin vector (5.21) becomes proportional to $\vec{k}$, such that the spin operator S_z approaches the helicity operator,

$$\begin{aligned} \hat{H}(\vec{k}) &= -\frac{1}{e_0} \not{k} \not{h} \gamma^5 \\ &= \hat{\vec{k}} \cdot \gamma^0 \vec{\gamma} \gamma^5, \qquad h^\mu = \frac{1}{e_0} \begin{pmatrix} |\vec{k}| \\ k_0 \hat{\vec{k}} \end{pmatrix} , \end{aligned} \tag{5.22}$$

as one would expect. As usually, the helicity operator measures spin in the direction of a particle's motion, $\hat{\vec{k}} = \vec{k}/|\vec{k}|$. As a consequence, for light particles with momenta of order the temperature, $k \sim T \gg m$, the spin states we consider here can be approximated by the helicity states, which are often used in literature for baryogenesis calculations.

The commutation relation (5.11) in the 1+1 frame now implies that by construction the spin operator $S_z(k)$ commutes with the kinetic differential operator $\mathcal{D}$ in (5.1), provided the bubble wall is stationary and x-y-symmetric in the wall frame. We can slightly relax this condition: S_z and $\mathcal{D}$ commute even for non-stationary and x-y-dependent bubble walls, as long as the dependence is of the form

$$S_s = S_s(k, t - \vec{v}_\parallel \cdot \vec{x}_\parallel, z)\,, \tag{5.23}$$

such that $\nabla_\parallel = -\vec{v}_\parallel \partial_t$ in $\mathcal{D}$. So we can treat time dependent problems, which can be used to study how the system relaxes from some initial conditions to the stationary state, admittedly only for quite special forms of the initial conditions.

As a consequence of the above discussion, even in the original moving frame, the problem splits into two non-mixing sectors labeled by the spin s:

$$S = \sum_{s=\pm 1} S_s\,, \qquad S_s \equiv P_s S P_s\,, \tag{5.24}$$

where $P_s(k) = (\mathbb{1}+sS_z(k))/2$, $s = \pm 1$ is the spin projector. We can again write down a decomposition of S_s,

$$S_s = iP_s\left[s\gamma^3\gamma^5 g_0^s - s\gamma^3 g_3^s + \mathbb{1}g_1^s - i\gamma^5 g_2^s\right] , \tag{5.25}$$

using a set of matrices that commute with $P_s(k)$. This is nothing else than the boosted 1+1 dimensional spin diagonal Wigner function:

$$\tilde{S}_s(\tilde{k},\tilde{x}) = L(k)S_s(k,x)L^{-1}(k) . \tag{5.26}$$

As a consequence of the hermiticity property (2.51), the scalar functions $g_a^s(k,x) = \tilde{g}_a^s(\tilde{k},\tilde{x})$ are all real. Since the direction in which spin is measured now depends on the momentum, it is not any more possible to find a representation of the Dirac algebra in which the block diagonality of S_s is explicitly displayed.

5.2. Constraint and kinetic equations

We are now ready to study the tree-level fermion dynamics governed by Eq. (5.1). In order to get the component equations, we take the block diagonal form (5.25) for the Wigner function S_s, multiply (5.1) by $P_s(k)\{1, s\gamma^3\gamma^5, -is\gamma^3, -\gamma^5\}$ and take the trace. The resulting equations are

$$\begin{aligned}
2i\hat{k}_0 g_0^s - 2is\hat{k}_z g_3^s - 2i\hat{m}_h g_1^s - 2i\hat{m}_a g_2^s &= \mathrm{Tr}P_s(k)\mathbb{1}\mathcal{C}_\psi && (5.27)\\
2i\hat{k}_0 g_1^s - 2s\hat{k}_z g_2^s - 2i\hat{m}_h g_0^s + 2\hat{m}_a g_3^s &= \mathrm{Tr}P_s(k)s\gamma^3\gamma^5\mathcal{C}_\psi && (5.28)\\
2i\hat{k}_0 g_2^s + 2s\hat{k}_z g_1^s - 2\hat{m}_h g_3^s - 2i\hat{m}_a g_0^s &= \mathrm{Tr}P_s(k)\left(-is\gamma^3\right)\mathcal{C}_\psi && (5.29)\\
2i\hat{k}_0 g_3^s - 2is\hat{k}_z g_0^s + 2\hat{m}_h g_2^s - 2\hat{m}_a g_1^s &= \mathrm{Tr}P_s(k)\left(-\gamma^5\right)\mathcal{C}_\psi , && (5.30)
\end{aligned}$$

where we used the shorthand notations

$$\hat{k}_0 = \tilde{k}_0 + \frac{i}{2}\frac{k_0\partial_t + \vec{k}_\parallel\cdot\nabla_\parallel}{\tilde{k}_0}, \qquad \hat{k}_z = k_z - \frac{i}{2}\partial_z , \tag{5.31}$$

and

$$\hat{m}_{h,a} = m_{h,a}(z)\mathrm{e}^{\frac{i}{2}\overleftarrow{\partial_z}\partial_{k_z}} . \tag{5.32}$$

For completeness, we have added the contributions from the fermionic collision term. One should keep in mind that the dependence of the Wigner function on the time and parallel coordinates is restricted by (5.23).

In the general case with fermionic mixing, each of the functions g_a^s is a hermitean matrix in flavor space. The mixing is mediated through the off-diagonal elements of the mass terms (5.32) and thus appears already at the leading order in gradient expansion; hence it would be inappropriate to work in the interaction basis (5.27-5.30) without incorporating the flavor off-diagonal elements. Since when integrated over the momenta $\int d^4k$ and $\int d^4k(k^\mu/k_0)$, equations (5.27-5.30) yield fluid equations, the same conclusions hold for fluid equations. This is in discord with the strategy advocated in a recent work [51] on chargino-mediated baryogenesis in the Minimal Supersymmetric Standard Model, where it was argued that, when considering the dynamics of CP-violating sources, one should work in the weak interaction basis for charginos (without taking account of the off-diagonals).

Rather than considering the full dynamics of mixing fermions, we shall now argue that, in order to properly capture the dynamics of CP-violating densities to order $\hbar$, it suffices to work in the spin and flavor diagonal bases, provided one transforms into the mass eigenbasis, in which m is diagonal. Here we generalize the analysis of Ref. [52] to 3+1 dimensions and include the space-time transients that accord with (5.23). More importantly, we fill in a gap in our original derivation in [52].

5.2.1. *Flavor diagonalization*

In order to get to the mass eigenbasis, we diagonalize the fermionic mass matrix m. Since m is in general nonhermitean, the diagonalization is exacted by the biunitary transformation

$$m_d = UmV^\dagger\,, \tag{5.33}$$

where U and V are the unitary matrices that diagonalize $mm^\dagger$ and $m^\dagger m$, respectively. To make the analysis more transparent, we make an explicit separation of the spinor and flavor spaces by using a direct product notation $\otimes$. The master equation (5.1) then becomes

$$\left(\not{k}\otimes\mathbb{1} + \frac{i}{2}\not{\partial}\otimes\mathbb{1} - \hat{\mathbf{m}}\right) S = \mathcal{C}_\psi, \tag{5.34}$$

where we have defined the mass term as

$$\begin{aligned}\hat{\mathbf{m}} &= P_R\otimes\hat{m} + P_L\otimes\hat{m}^\dagger\\ &= \mathbb{1}\otimes\hat{m}_h + i\gamma^5\otimes\hat{m}_a\,.\end{aligned} \tag{5.35}$$

It is now a simple matter to see that the mass matrix $\mathbf{m}$ is diagonalized by the unitary matrices

$$\begin{aligned}\mathbf{X} &= P_L\otimes V + P_R\otimes U = \mathbb{1}\otimes\frac{1}{2}(V+U) - \gamma^5\otimes\frac{1}{2}(V-U)\\ \mathbf{Y} &= P_L\otimes U + P_R\otimes V = \mathbb{1}\otimes\frac{1}{2}(V+U) + \gamma^5\otimes\frac{1}{2}(V-U)\end{aligned} \tag{5.36}$$

as follows:

$$\mathbf{m}_d = \mathbf{X}\mathbf{m}\mathbf{Y}^\dagger\,. \tag{5.37}$$

The Wigner function then transforms as

$$S_d = \mathbf{Y}S\mathbf{X}^\dagger\,, \tag{5.38}$$

which is in general not diagonal. From here on we work in the frame where the bubble wall is at rest, so that we can make a spin-diagonal ansatz for the rotated Wigner function S_{sd} which is analogous to (5.25). Note that the spin projector P_s commutes with the rotation matrices. Since the mass term $\hat{\mathbf{m}}$ mixes spinor and flavor, the rotation matrices $\mathbf{Y}$ and $\mathbf{X}$ do so as well. As a consequence, the component functions g^s_{ad} of the rotated Wigner function are not just the flavor rotated components g^s_a of S_s. Indeed, inserting (5.25) into (5.38) leads to the following relations

$$g^s_{0d} = \frac{1}{2}\left[V(g^s_0+g^s_3)V^\dagger + U(g^s_0-g^s_3)U^\dagger\right] \tag{5.39}$$

$$g^s_{3d} = \frac{1}{2}\left[V(g^s_3+g^s_0)V^\dagger - U(g^s_0-g^s_3)U^\dagger\right] \tag{5.40}$$

$$g^s_{1d} = \frac{1}{2}\left[V(g^s_1-ig^s_2)U^\dagger + U(g^s_1+ig^s_2)V^\dagger\right] \tag{5.41}$$

$$g^s_{2d} = \frac{1}{2}\left[V(g^s_2+ig^s_1)U^\dagger + U(g^s_2-ig^s_1)V^\dagger\right]\,, \tag{5.42}$$

that is g^s_0 mixes with g^s_3 and g^s_1 mixes with g^s_2. We now transform (5.34) by multiplying it from the left by $\mathbf{X}$ and from the right by $\mathbf{X}^\dagger$ to obtain

$$\left(\not{k}\otimes\mathbb{1}+\frac{i}{2}\not{\mathbf{D}}-\mathbf{m}_d\,\mathrm{e}^{\frac{i}{2}\overleftarrow{\mathbf{D}}_z\partial_{k_z}}\right)\;S_d=\mathcal{C}_{\psi d}\,, \tag{5.43}$$

where $\mathcal{C}_{\psi d}=\mathbf{X}\mathcal{C}_\psi\mathbf{X}^\dagger$, and we defined a 'covariant' derivative

$$\mathbf{D}_\mu=\mathbb{1}\otimes\mathbb{1}\partial_\mu-i[\mathbb{1}\otimes\Sigma_\mu,\cdot]-i\{\gamma^5\otimes\Delta_\mu,\cdot\}\,, \tag{5.44}$$

where Σ_μ and Δ_μ are given in terms of the rotation matrices U and V as follows:

$$\Delta_\mu\;=\;\frac{i}{2}\left(V\partial_\mu V^\dagger-U\partial_\mu U^\dagger\right) \tag{5.45}$$

$$\Sigma_\mu\;=\;\frac{i}{2}\left(V\partial_\mu V^\dagger+U\partial_\mu U^\dagger\right)\,. \tag{5.46}$$

Note that the covariant derivative (5.44) can be obtained from

$$\partial_\mu\mathbf{m}=\partial_\mu(\mathbf{X}^\dagger\mathbf{m}_d\mathbf{Y})=\mathbf{X}^\dagger(\mathbf{D}_\mu\mathbf{m}_d)\mathbf{Y} \tag{5.47}$$

The higher order derivatives are obtained simply by iteration. Already from (5.43-5.44) it is clear that the kinetic and constraint equations, when written in the mass eigenbasis, will contain extra commutator and anticommutator terms. The choice of the rotation matrices U and V is not unique. After an x-dependent phase redefinition $U\to wU$ and $V\to wV$, where w is a diagonal matrix with eigenvalues of absolute value 1, U and V still diagonalize m. This freedom to redefine the rotation matrices was the source of some problems in finding the correct physical source in the WKB approach. We will find, however, that only the diagonal elements of the matrix Δ_μ are of relevance, and these are invariant under this reparametrization.
We now project out the spinor structure in (5.43) by multiplying by $P_s(k)\{1,s\gamma^3\gamma^5,-is\gamma^3,-\gamma^5\}$ and taking the spinorial traces. The procedure is identical to the derivation of (5.27-5.30) at the beginning of section 5.2, except for the subtlety related to the covariant derivative, because it contains spinor structure. The resulting equations are

$$\left(2i\tilde{k}_0-\mathcal{D}_t^-\right)g^s_{0d}-s\left(2ik_z+\mathcal{D}_z\right)g^s_{3d}-2im_{hd}\,\mathrm{e}^{\frac{i}{2}\overleftarrow{D}_z\partial_{k_z}}g^s_{1d}-2im_{ad}\mathrm{e}^{\frac{i}{2}\overleftarrow{D}_z\partial_{k_z}}g^s_{2d}=\mathrm{Tr}\mathbb{1}P_s\mathcal{C}_{\psi d} \tag{5.48}$$

$$\left(2i\tilde{k}_0-\mathcal{D}_t^+\right)g^s_{1d}-s\left(2k_z-i\mathcal{D}_z\right)g^s_{2d}-2im_{hd}\mathrm{e}^{\frac{i}{2}\overleftarrow{D}_z\partial_{k_z}}g^s_{0d}+2m_{ad}\mathrm{e}^{-\frac{i}{2}\overleftarrow{D}_z\partial_{k_z}}g^s_{3d}=\mathrm{Tr}(s\gamma^3\gamma^5)P_s\mathcal{C}_{\psi d} \tag{5.49}$$

$$\left(2i\tilde{k}_0-\mathcal{D}_t^+\right)g^s_{2d}+s\left(2k_z-i\mathcal{D}_z\right)g^s_{1d}-2m_{hd}\mathrm{e}^{\frac{i}{2}\overleftarrow{D}_z\partial_{k_z}}g^s_{3d}-2im_{ad}\mathrm{e}^{\frac{i}{2}\overleftarrow{D}_z\partial_{k_z}}g^s_{0d}=\mathrm{Tr}\left(-is\gamma^3\right)P_s\mathcal{C}_{\psi d} \tag{5.50}$$

$$\left(2i\tilde{k}_0-\mathcal{D}_t^-\right)g^s_{3d}-s\left(2ik_z+\mathcal{D}_z\right)g^s_{0d}+2m_{hd}\mathrm{e}^{\frac{i}{2}\overleftarrow{D}_z\partial_{k_z}}g^s_{2d}-2m_{ad}\mathrm{e}^{\frac{i}{2}\overleftarrow{D}_z\partial_{k_z}}g^s_{1d}=\mathrm{Tr}\left(-\gamma^5\right)P_s\mathcal{C}_{\psi d}\,, \tag{5.51}$$

where we have defined the derivatives

$$\begin{aligned}D_zm_{hd}\;&\equiv\;\partial_zm_{hd}-i[\Sigma_z,m_{hd}]-\{\Delta_z,m_{ad}\}\\ D_zm_{ad}\;&\equiv\;\partial_zm_{ad}-i[\Sigma_z,m_{ad}]+\{\Delta_z,m_{hd}\}\end{aligned} \tag{5.52}$$

and

$$\mathcal{D}_t^- = \gamma_\| \partial_t + \gamma_\| \vec{v}_\| \cdot \nabla_\| - is[\Delta_z, \cdot] \tag{5.53}$$
$$\mathcal{D}_t^+ = \gamma_\| \partial_t + \gamma_\| \vec{v}_\| \cdot \nabla_\| - is\{\Delta_z, \cdot\} \tag{5.54}$$
$$\mathcal{D}_z = \partial_z - i[\Sigma_z, \cdot] \tag{5.55}$$

Note that due to the γ^5 the anticommutator in (5.44) has become the commutator in (5.53). Moreover, the derivative $\mathcal{D}_t^+$ is not hermitean. Indeed, the anticommutator term in (5.54) is antihermitean.

5.2.2. *Constraint equations*

The constraint equations correspond to the antihermitean parts of the equations (5.48)-(5.51). Since we are interested in the dispersion relation accurate to first order in $\hbar$, it suffices to consider these equations to first order in gradients:

$$2\tilde{k}_0 g_{0d}^s - 2sk_z g_{3d}^s - \big\{m_{hd}, g_{1d}^s\big\} - \frac{i}{2}\big[D_z m_{hd}, \partial_{k_z} g_{1d}^s\big] - \big\{m_{ad}, g_{2d}^s\big\} - \frac{i}{2}\big[D_z m_{ad}, \partial_{k_z} g_{2d}^s\big] = \mathcal{C}_{0d}^s \tag{5.56}$$

$$\begin{aligned} 2\tilde{k}_0 g_{1d}^s + s\{\Delta_z, g_{1d}^s\} + s(\partial_z - i[\Sigma_z, \cdot])g_{2d}^s - \big\{m_{hd}, g_{0d}^s\big\} - \frac{i}{2}\big[D_z m_{hd}, \partial_{k_z} g_{0d}^s\big] & \\ + \frac{1}{2}\big\{D_z m_{ad}, \partial_{k_z} g_{3d}^s\big\} - i\big[m_{ad}, g_{3d}^s\big] &= \mathcal{C}_{1d}^s \end{aligned} \tag{5.57}$$

$$\begin{aligned} 2\tilde{k}_0 g_{2d}^s + s\{\Delta_z, g_{2d}^s\} - s(\partial_z - i[\Sigma_z, \cdot])g_{1d}^s - \frac{1}{2}\big\{D_z m_{hd}, \partial_{k_z} g_{3d}^s\big\} + i\big[m_{hd}, g_{3d}^s\big] & \\ - \big\{m_{ad}, g_{0d}^s\big\} - \frac{i}{2}\big[D_z m_{ad}, \partial_{k_z} g_{0d}^s\big] &= \mathcal{C}_{2d}^s \end{aligned} \tag{5.58}$$

$$2\tilde{k}_0 g_{3d}^s - 2sk_z g_{0d}^s + \frac{1}{2}\big\{D_z m_{hd}, \partial_{k_z} g_{2d}^s\big\} - i\big[m_{hd}, g_{2d}^s\big] - \frac{1}{2}\big\{D_z m_{ad}, \partial_{k_z} g_{1d}^s\big\} + i\big[m_{ad}, g_{1d}^s\big] = \mathcal{C}_{3d}^s\,, \tag{5.59}$$

where

$$\begin{aligned} \mathcal{C}_{0d}^s &= \frac{1}{2i}\,\mathrm{Tr}\mathbb{1}\Big(P_s(k)\mathcal{C}_{\psi d} - P_s^\dagger(k)\mathcal{C}_{\psi d}^\dagger\Big) \\ \mathcal{C}_{1d}^s &= \frac{1}{2i}\,\mathrm{Tr}(s\gamma^3\gamma^5)\Big(P_s(k)\mathcal{C}_{\psi d} - P_s^\dagger(k)\mathcal{C}_{\psi d}^\dagger\Big) \\ \mathcal{C}_{2d}^s &= \frac{1}{2i}\,\mathrm{Tr}(-is\gamma^3)\Big(P_s(k)\mathcal{C}_{\psi d} - P_s^\dagger(k)\mathcal{C}_{\psi d}^\dagger\Big) \\ \mathcal{C}_{3d}^s &= \frac{1}{2i}\,\mathrm{Tr}(-\gamma^5)\Big(P_s(k)\mathcal{C}_{\psi d} - P_s^\dagger(k)\mathcal{C}_{\psi d}^\dagger\Big)\,. \end{aligned} \tag{5.60}$$

The trace is only to be taken in spinor space, not in the fermionic flavor space. For simplicity we now perform the analysis for two mixing fermions. Our findings are however valid for an arbitrary number of mixing fermions. Taking account of the fact that the off-diagonals are of the order $\hbar$, equations (5.56-5.59) now imply the following diagonal equations accurate to order $\hbar$,

$$2\tilde{k}_0\, g_{0d11}^s - 2m_{hd1}\, g_{1d11}^s - 2m_{ad1}\, g_{2d11}^s - 2sk_z\, g_{3d11}^s = \mathcal{C}_{0d11}^s \tag{5.61}$$
$$-2m_{hd1}\, g_{0d11}^s + 2\tilde{k}_0\, g_{1d11}^s + 2s\Delta_{z11}\, g_{1d11}^s + s\partial_z\, g_{2d11}^s + (D_z m_{ad})_{11}\partial_{k_z} g_{3d11}^s = \mathcal{C}_{1d11}^s \tag{5.62}$$
$$-2m_{ad1}\, g_{0d11}^s - s\partial_z\, g_{1d11}^s + 2\tilde{k}_0\, g_{2d11}^s + 2s\Delta_{z11}\, g_{2d11}^s - (D_z m_{hd})_{11}\partial_{k_z} g_{3d11}^s = \mathcal{C}_{1d11}^s \tag{5.63}$$
$$-2sk_z\, g_{0d11}^s - (D_z m_{ad})_{11}\partial_{k_z} g_{1d11}^s + (D_z m_{hd})_{11}\partial_{k_z} g_{2d11}^s + 2\tilde{k}_0\, g_{3d11}^s = \mathcal{C}_{3d11}^s\,, \tag{5.64}$$

and similarly for the (22)-components. Solving the latter three equations in gradient expansion in terms of g^s_{0d11} we find

$$
\begin{aligned}
g^s_{1d11} &= \frac{1}{\tilde{k}_0}\Big[m_{hd1}\, g^s_{0d11} - \frac{1}{\tilde{k}_0} s\Delta_{z11}\,(m_{hd1}\, g^s_{0d11}) - \frac{1}{2\tilde{k}_0} s\partial_z\,(m_{ad1}\, g^s_{0d11}) \\
&\qquad - \frac{s}{2\tilde{k}_0}(D_z m_{ad})_{11}(1 + k_z\partial_{k_z}) g^s_{0d11} + \frac{1}{2}\mathcal{C}^s_{1d11}\Big] \qquad (5.65)\\
g^s_{2d11} &= \frac{1}{\tilde{k}_0}\Big[m_{ad1}\, g^s_{0d11} + \frac{1}{2\tilde{k}_0} s\partial_z\,(m_{hd1} g^s_{0d11}) - \frac{1}{\tilde{k}_0} s\Delta_{z11}(m_{ad1} g^s_{0d11}) \\
&\qquad + \frac{s}{2\tilde{k}_0}(D_z m_{hd})_{11}(1 + k_z\partial_{k_z}) g^s_{0d11} + \frac{1}{2}\mathcal{C}^s_{2d11}\Big] \qquad (5.66)\\
g^s_{3d11} &= \frac{1}{\tilde{k}_0}\Big[s k_z\, g^s_{0d11} + \frac{1}{2\tilde{k}_0} m_{hd1}(D_z m_{ad})_{11}\partial_{k_z} g^s_{0d11} - \frac{1}{2\tilde{k}_0} m_{ad1}(D_z m_{hd})_{11}\partial_{k_z} g^s_{0d11} + \frac{1}{2}\mathcal{C}^s_{3d11}\Big]\,. \qquad (5.67)
\end{aligned}
$$

Here m_{hd1} denotes the first diagonal element of m_{hd}, m_{ad1} is defined correspondingly. Inserting these relations into (5.61) we get the constraint for the diagonal densities

$$
\begin{aligned}
&\frac{2}{\tilde{k}_0}\Big(k^2 - |m_d|^2_i + \frac{s}{\tilde{k}_0}\big[|m_d|^2_i(\partial_z\theta_{di} + 2\Delta_{z\mathtt{ii}})\big]\Big)\, g^s_{0d\mathtt{ii}} \\
&\qquad = \mathcal{C}^s_{0d\mathtt{ii}} + \frac{1}{\tilde{k}_0}\big(m_{hdi}\mathcal{C}^s_{1d\mathtt{ii}} + m_{adi}\mathcal{C}^s_{2d\mathtt{ii}} + s k_z \mathcal{C}^s_{3d\mathtt{ii}}\big)\,, \qquad (5.68)
\end{aligned}
$$

where we defined

$$
\begin{aligned}
|m_d|^2_i &= m_{hd}{}^2_i + m_{ad}{}^2_i \\
|m_d|^2_i\partial_z\theta_{di} &= m_{hdi}\partial_z m_{adi} - m_{adi}\partial_z m_{hdi}\,, \qquad (5.69)
\end{aligned}
$$

where $i = 1, 2$. Note that the energy shift in (5.68) can be written in terms of the rotation matrices U as

$$
|m_d|^2_i(\partial_z\theta_{di} + 2\Delta_{z\mathtt{ii}}) = -\Im[U(m\partial_z m^\dagger)U^\dagger]_{\mathtt{ii}}\,. \qquad (5.70)
$$

5.2.3. Quasiparticle picture and dispersion relation

Equation (5.68) specifies the spectral properties of the plasma excitations. Since it is an algebraic constraint, the *quasiparticle picture* remains a valid description of the plasma to order $\hbar$, as it was already pointed out in [52, 53]. Since the constraint equation contains no time dependence, it measures genuine spectral properties independent of the actual dynamical populations of the states. The homogeneous solution of the constraint equation (5.68) has the following spectral form

$$
\begin{aligned}
g^{<s}_{0d\,\mathtt{ii}}(k,x) &= 2\pi\delta\Big(k^2 - |m_d|^2_i + \frac{s}{\tilde{k}_0}\big[|m_d|^2_i(\partial_z\theta_{di} + 2\Delta_{z\mathtt{ii}})\big]\Big)|\tilde{k}_0|\, n_{si}(k,x) \\
&= 2\pi\sum_{\pm}\frac{\delta(k_0 \mp \omega_{\pm si})}{2\omega_{\pm si}Z_{\pm si}}\,|\tilde{k}_0|\, n_{si}(k,x)\,, \qquad (5.71)
\end{aligned}
$$

with the following *dispersion relations* for fermions

$$
\omega_{si} = \omega_{0i} - \frac{s}{2\tilde{\omega}_{0i}\omega_{0i}}\big[|m_d|^2_i(\partial_z\theta_{di} + 2\Delta_{z\mathtt{ii}})\big], \qquad (5.72)
$$

where

$$\begin{aligned} \omega_{0i} &= (\vec{k}^2 + |m_d|_i^2)^{\frac{1}{2}} \\ \tilde{\omega}_{0i} &= (k_z^2 + |m_d|_i^2)^{\frac{1}{2}} . \end{aligned} \tag{5.73}$$

The normalization factors are

$$Z_{si} = 1 - \frac{s|m_d|_i^2(\partial_z\theta_{di} + 2\Delta_{z\mathtt{ii}})}{2\tilde{\omega}_{0i}^3} . \tag{5.74}$$

We have normalized the solution (5.71) such that $n_{si}(k,x)$ represents the particle density in phase space $\{k,x\}$. We emphasize that the spectral solution (5.71) and the dispersion relation (5.72) are valid in general for all plasma excitations: equilibrium and stationary excitations, as well as for non-equilibrium, time dependent transients that conserve spin, that is for $g_{0d}^s = g_{0d}^s(k, t - \vec{k}_\parallel \cdot \vec{x}_\parallel, z)$, assuming of course planar symmetry in the wall frame.
Since in the on-shell limit the Wigner functions $S^<$ and $S^>$ satisfy identical Kadanoff-Baym equations (2.55), and in addition they are related by the fermionic sum rule (2.60), based on the above analysis of $S^<$ we can easily reconstruct the spectral solution for the Wigner function $S^>$, which we write as

$$S_d^> = \sum_{s=\pm1} iP_s(k)\left[s\gamma^3\gamma^5 g_{0d}^{>s} - s\gamma^3 g_{3d}^{>s} + \mathbb{1} g_{1d}^{>s} - i\gamma^5 g_{2d}^{>s}\right] , \tag{5.75}$$

where the spectral solution for $g_{0d}^{>s}$ reads

$$g_{0d\mathtt{ii}}^{>s}(k,x) = -2\pi\sum_{\pm}\frac{\delta(k_0 \mp \omega_{\pm si})}{2\omega_{\pm si}Z_{\pm si}}|\tilde{k}_0|\left[1 - n_{si}(k,x)\right] . \tag{5.76}$$

Note that $g_{0d\mathtt{ii}}^{>s}$ obeys the same equation (5.68) as $g_{0d\mathtt{ii}}^{<s}$. Similarly, $g_{1d\mathtt{ii}}^{>s}$, $g_{2d\mathtt{ii}}^{>s}$ and $g_{3d\mathtt{ii}}^{>s}$ are related to $g_{0d\mathtt{ii}}^{>s}$ the same way as indicated in Eqs. (5.65-5.67).

5.2.4. Plasma frame

An interesting question is of course how to generalize the solution for the Wigner function (5.25) to other Lorentz frames, an important example being the plasma rest frame. The natural solution is to apply a Lorentz boost on the Wigner function. The simplest boost operator corresponding to a boost v in z-direction can be easily constructed in analogy with (5.4), and it reads

$$L(\Lambda_z) = \frac{\gamma + 1 - \gamma v \cdot \gamma^0\gamma^3}{[2(\gamma+1)]^{1/2}}, \tag{5.77}$$

where $\gamma = (1-v^2)^{-1/2}$. The Wigner function then transforms as

$$\begin{aligned} \breve{S}_s &= L^{-1}(\Lambda_z)S_sL(\Lambda_z) \\ &= -\breve{P}_s(k)\left[\gamma(1+v\gamma^0\gamma^3)\left(s\gamma^3\gamma^5\tilde{g}_0^s - s\gamma^3\tilde{g}_3^s\right) + \mathbb{1}\tilde{g}_1^s - i\gamma^5\tilde{g}_2^s\right] , \end{aligned} \tag{5.78}$$

where

$$\breve{P}_s(\breve{k}) = \frac{\gamma(\breve{k}_0 - v\breve{k}_z)}{\breve{\tilde{k}}_0}\gamma^0\gamma^3\gamma^5 - \gamma(1+v\gamma^0\gamma^3)\frac{\breve{\vec{k}}_\parallel}{\breve{\tilde{k}}_0}\cdot\vec{\gamma}_\parallel\gamma^3\gamma^5 \tag{5.79}$$

is the spin operator with $\breve{\tilde{k}}_0^2 = \gamma^2(\breve{k}_0 - v\breve{k}_z)^2 - \breve{\vec{k}}_\parallel^2$, and $g_a^s = g_a^s(\breve{k}, \breve{x})$. We leave as an exercise to the reader to find the direction of the spin vector $\breve{s}^\mu$ corresponding to the spin operator (5.79). To get the final form for the Wigner function in the new frame, one also needs to transform the coordinates and derivatives in (5.65-5.67) and (5.71), which makes the final expression for $i\breve{S}_s^<$ rather cumbersome. To get the Wigner function $i\breve{S}_s$ in the *plasma frame*, one needs to choose the boost that corresponds to $v = v_w$ and $\gamma = \gamma_w = (1 - v_w^2)^{-1/2}$. This exercise underlines clearly the advantages of working in the wall frame, in which the Wigner function has a particularly simple form. Some authors [50, 51] choose nevertheless to work in the plasma frame, although in a simplistic disguise.

5.2.5. *Kinetic equations*

We now take (minus) the hermitean part of Eqs. (5.48-5.51) and get the following *kinetic equations*

$$\begin{aligned}
&\mathcal{D}_t^- g_{0d}^s + s\mathcal{D}_z g_{3d}^s - \{m_{hd}\sin\big(\tfrac{1}{2}\overleftarrow{D}_z\partial_{k_z}\big), g_{1d}^s\} + i[m_{hd}\cos\big(\tfrac{1}{2}\overleftarrow{D}_z\partial_{k_z}\big), g_{1d}^s] \\
&\qquad - \{m_{ad}\sin\big(\tfrac{1}{2}\overleftarrow{D}_z\partial_{k_z}\big), g_{2d}^s\} + i[m_{ad}\cos\big(\tfrac{1}{2}\overleftarrow{D}_z\partial_{k_z}\big), g_{2d}^s] = \mathcal{K}_{0d}^s \qquad (5.80) \\
&(\mathcal{D}_t^+)_h g_{1d}^s + 2sk_z g_{2d}^s - \{m_{hd}\sin\big(\tfrac{1}{2}\overleftarrow{D}_z\partial_{k_z}\big), g_{0d}^s\} + i[m_{hd}\cos\big(\tfrac{1}{2}\overleftarrow{D}_z\partial_{k_z}\big), g_{0d}^s] \\
&\qquad - \{m_{ad}\cos\big(\tfrac{1}{2}\overleftarrow{D}_z\partial_{k_z}\big), g_{3d}^s\} - i[m_{ad}\sin\big(\tfrac{1}{2}\overleftarrow{D}_z\partial_{k_z}\big), g_{3d}^s] = \mathcal{K}_{1d}^s \qquad (5.81) \\
&(\mathcal{D}_t^+)_h g_{2d}^s - 2sk_z g_{1d}^s + \{m_{hd}\cos\big(\tfrac{1}{2}\overleftarrow{D}_z\partial_{k_z}\big), g_{3d}^s\} + i[m_{hd}\sin\big(\tfrac{1}{2}\overleftarrow{D}_z\partial_{k_z}\big), g_{3d}^s] \\
&\qquad - \{m_{ad}\sin\big(\tfrac{1}{2}\overleftarrow{D}_z\partial_{k_z}\big), g_{0d}^s\} + i[m_{ad}\cos\big(\tfrac{1}{2}\overleftarrow{D}_z\partial_{k_z}\big), g_{0d}^s] = \mathcal{K}_{2d}^s \qquad (5.82) \\
&\mathcal{D}_t^- g_{3d}^s + s\mathcal{D}_z g_{0d}^s - \{m_{hd}\cos\big(\tfrac{1}{2}\overleftarrow{D}_z\partial_{k_z}\big), g_{2d}^s\} - i[m_{hd}\sin\big(\tfrac{1}{2}\overleftarrow{D}_z\partial_{k_z}\big), g_{2d}^s] \\
&\qquad + \{m_{ad}\cos\big(\tfrac{1}{2}\overleftarrow{D}_z\partial_{k_z}\big), g_{1d}^s\} + i[m_{ad}\sin\big(\tfrac{1}{2}\overleftarrow{D}_z\partial_{k_z}\big), g_{1d}^s] = \mathcal{K}_{3d}^s\,, \qquad (5.83)
\end{aligned}$$

where $(\mathcal{D}_t^+)_h = \gamma_\parallel(\partial_t - i[\Sigma_t, \cdot]) + \gamma_\parallel\vec{v}_\parallel\cdot\nabla_\parallel$ denotes the hermitean part of the derivative, and

$$\begin{aligned}
\mathcal{K}_{0d}^s &= -\frac{1}{2}\,\mathrm{Tr}\mathbb{1}\Big(P_s(k)\mathcal{C}_{\psi d} + P_s^\dagger(k)\mathcal{C}_{\psi d}^\dagger\Big) \\
\mathcal{K}_{1d}^s &= -\frac{1}{2}\,\mathrm{Tr}(s\gamma^3\gamma^5)\Big(P_s(k)\mathcal{C}_{\psi d} + P_s^\dagger(k)\mathcal{C}_{\psi d}^\dagger\Big) \\
\mathcal{K}_{2d}^s &= -\frac{1}{2}\,\mathrm{Tr}(-is\gamma^3)\Big(P_s(k)\mathcal{C}_{\psi d} + P_s^\dagger(k)\mathcal{C}_{\psi d}^\dagger\Big) \\
\mathcal{K}_{3d}^s &= -\frac{1}{2}\,\mathrm{Tr}(-\gamma^5)\Big(P_s(k)\mathcal{C}_{\psi d} + P_s^\dagger(k)\mathcal{C}_{\psi d}^\dagger\Big)\,. \qquad (5.84)
\end{aligned}$$

Again the trace must only be taken in spinor space. Since we are interested in the order $\hbar$ effects, we can truncate the kinetic equations (5.80-5.83) to second order in gradients. For example, the kinetic equation for particle density of spin s (5.80) reads

$$\begin{aligned}
&(\gamma_\parallel\partial_t + \gamma_\parallel\vec{v}_\parallel\cdot\nabla_\parallel - is[\Delta_z, \cdot])g_{0d}^s + s(\partial_z - i[\Sigma_z, \cdot])g_{3d}^s \\
&\qquad - \frac{1}{2}\{D_z m_{hd}, \partial_{k_z} g_{1d}^s\} + i[m_{hd}, g_{1d}^s] - \frac{1}{2}\{D_z m_{ad}, \partial_{k_z} g_{2d}^s\} + i[m_{ad}, g_{2d}^s] = \mathcal{K}_{0d}^s\,, \qquad (5.85)
\end{aligned}$$

where we dropped the second order terms in the commutator, which is legitimate since they contribute only to the off-diagonal equations, and hence to order $\hbar^2$.
From (5.85) we can immediately write the kinetic equation for the diagonal densities

$$\begin{aligned}
& \gamma_\parallel(\partial_t + \vec{v}_\parallel \cdot \nabla_\parallel) g^s_{0d11} + s\partial_z g^s_{3d11} - (D_z m_{hd})_{11}\partial_{k_z} g^s_{1d11} - (D_z m_{ad})_{11}\partial_{k_z} g^s_{2d11} \\
\mp\ & is(\Sigma_{z12} g^s_{3d21} - \Sigma_{z21} g^s_{3d12} + \Delta_{z12} g^s_{0d21} - \Delta_{z21} g^s_{0d12}) \\
-\ & \frac{1}{2}\Big[(D_z m_{hd})_{12}\partial_{k_z} g^s_{1d21} + (D_z m_{hd})_{21}\partial_{k_z} g^s_{1d12} + (D_z m_{ad})_{12}\partial_{k_z} g^s_{2d21} + (D_z m_{ad})_{21}\partial_{k_z} g^s_{2d12}\Big] = \mathcal{K}^s_{0d11}.
\end{aligned} \tag{5.86}$$

In order to decouple the diagonal and off-diagonal equations we make use of the off-diagonal equations, which to leading order in gradients are

$$\begin{aligned}
& i\delta(m_{hd}) g^s_{1d12} + i\delta(m_{ad}) g^s_{2d12} \\
& \quad = -is\Delta_{z12}\delta(g^s_{0d}) + \frac{1}{2}(D_z m_{hd})_{12}\partial_{k_z}\mathrm{Tr}(g^s_{1d}) + \frac{1}{2}(D_z m_{ad})_{12}\partial_{k_z}\mathrm{Tr}(g^s_{2d}) - is\Sigma_{z12}\delta(g^s_{3d}) + \mathcal{K}^s_{0d12}
\end{aligned} \tag{5.87}$$

$$\begin{aligned}
& i\delta(m_{hd}) g^s_{0d12} + 2sk_z g^s_{2d12} - \mathrm{Tr}(m_{ad}) g^s_{3d12} \\
& \quad = \frac{1}{2}(D_z m_{hd})_{12}\partial_{k_z}\mathrm{Tr}(g^s_{0d}) - \frac{i}{2}(D_z m_{ad})_{12}\partial_{k_z}\delta(g^s_{3d}) + \mathcal{K}^s_{1d12}
\end{aligned} \tag{5.88}$$

$$\begin{aligned}
& i\delta(m_{ad}) g^s_{0d12} - 2sk_z g^s_{1d12} + \mathrm{Tr}(m_{hd}) g^s_{3d12} \\
& \quad = \frac{1}{2}(D_z m_{ad})_{12}\partial_{k_z}\mathrm{Tr}(g^s_{0d}) + \frac{i}{2}(D_z m_{hd})_{12}\partial_{k_z}\delta(g^s_{3d}) + \mathcal{K}^s_{2d12}
\end{aligned} \tag{5.89}$$

$$\begin{aligned}
& \mathrm{Tr}(m_{ad}) g^s_{1d12} - \mathrm{Tr}(m_{hd}) g^s_{2d12} \\
& \quad = -is\Sigma_{z12}\delta(g^s_{0d}) + \frac{i}{2}(D_z m_{ad})_{12}\partial_{k_z}\delta(g^s_{1d}) - \frac{i}{2}(D_z m_{hd})_{12}\partial_{k_z}\delta(g^s_{2d}) - is\Delta_{z12}\delta(g^s_{3d}) + \mathcal{K}^s_{3d12}\,.
\end{aligned} \tag{5.90}$$

Combining (5.87) and (5.90) we obtain

$$\begin{aligned}
g^s_{1d12} = \frac{1}{\delta(|m_d|)^2}\Big\{ & -s(D_z m_{ad})_{12}\delta(g^s_{0d}) - is[\Delta_{z12}\delta(m_{ad}) - i\Sigma_{z12}\mathrm{Tr}(m_{hd})]\delta(g^s_{3d}) \\
& - \frac{i}{2}[\mathrm{Tr}(m_{hd})(D_z m_{hd})_{12}\partial_{k_z}\mathrm{Tr}(g^s_{1d}) - \mathrm{Tr}(m_{ad})(D_z m_{ad})_{12}\partial_{k_z}\delta(g^s_{1d})] \\
& - \frac{i}{2}[\mathrm{Tr}(m_{hd})(D_z m_{ad})_{12}\partial_{k_z}\mathrm{Tr}(g^s_{2d}) + \mathrm{Tr}(m_{ad})(D_z m_{hd})_{12}\partial_{k_z}\delta(g^s_{2d})] \\
& - i\mathrm{Tr}(m_{hd})\mathcal{K}^s_{0d12} + \delta(m_{ad})\mathcal{K}^s_{3d12}\Big\}
\end{aligned} \tag{5.91}$$

$$\begin{aligned}
g^s_{2d12} = \frac{1}{\delta(|m_d|)^2}\Big\{ & \ \ s(D_z m_{hd})_{12}\delta(g^s_{0d}) + is[\Delta_{z12}\delta(m_{hd}) + i\Sigma_{z12}\mathrm{Tr}(m_{ad})]\delta(g^s_{3d}) \\
& - \frac{i}{2}[\mathrm{Tr}(m_{ad})(D_z m_{hd})_{12}\partial_{k_z}\mathrm{Tr}(g^s_{1d}) + \mathrm{Tr}(m_{hd})(D_z m_{ad})_{12}\partial_{k_z}\delta(g^s_{1d})] \\
& - \frac{i}{2}[\mathrm{Tr}(m_{ad})(D_z m_{ad})_{12}\partial_{k_z}\mathrm{Tr}(g^s_{2d}) - \mathrm{Tr}(m_{hd})(D_z m_{hd})_{12}\partial_{k_z}\delta(g^s_{2d})] \\
& - i\mathrm{Tr}(m_{ad})\mathcal{K}^s_{0d12} - \delta(m_{hd})\mathcal{K}^s_{3d12}\Big\}\,,
\end{aligned} \tag{5.92}$$

where

$$\delta(|m_d|)^2 = \mathrm{Tr}(m_{hd})\delta(m_{hd}) + \mathrm{Tr}(m_{ad})\delta(m_{ad})\,. \tag{5.93}$$

From (5.87-5.90) and (5.91-5.92) we then get

$$
\begin{aligned}
g^s_{0d12} \;=\; \frac{1}{\delta(|m_d|)^2}\Big\{ & -2k_z[\Sigma_{z12}\delta(g^s_{0d}) + \Delta_{z12}\delta(g^s_{3d})] + \frac{1}{2}\Sigma_{z12}\delta(|m_d|^2)\partial_{k_z}\mathrm{Tr}(g^s_{0d}) \\
& + sk_z[(D_z m_{ad})_{12}\partial_{k_z}\delta(g^s_{1d}) - (D_z m_{hd})_{12}\partial_{k_z}\delta(g^s_{2d})] \\
& - \frac{1}{2}[\mathrm{Tr}(m_{hd})(D_z m_{ad})_{12} - \mathrm{Tr}(m_{ad})(D_z m_{hd})_{12}]\partial_{k_z}\delta(g^s_{3d}) \\
& - i\mathrm{Tr}(m_{hd})\mathcal{K}^s_{1d12} - i\mathrm{Tr}(m_{ad})\mathcal{K}^s_{2d12} - 2isk_z\mathcal{K}^s_{3d12}\Big\}
\end{aligned} \tag{5.94}
$$

$$
\begin{aligned}
g^s_{3d12} \;=\; \frac{1}{\delta(|m_d|)^2}\Big\{ & -2k_z[\Delta_{z12}\delta(g^s_{0d}) + \Sigma_{z12}\delta(g^s_{3d})] + \frac{1}{2}\Delta_{z12}\delta(|m_d|^2)\partial_{k_z}\mathrm{Tr}(g^s_{0d}) \\
& - isk_z[(D_z m_{hd})_{12}\partial_{k_z}\mathrm{Tr}(g^s_{1d}) + (D_z m_{ad})_{12}\partial_{k_z}\mathrm{Tr}(g^s_{2d})] \\
& + \frac{i}{2}[\delta(m_{ad})(D_z m_{ad})_{12} + \delta(m_{hd})(D_z m_{hd})_{12}]\partial_{k_z}\delta(g^s_{3d}) \\
& - 2isk_z\mathcal{K}^s_{0d12} - \delta(m_{ad})\mathcal{K}^s_{1d12} - \delta(m_{hd})\mathcal{K}^s_{2d12}\Big\}\,.
\end{aligned} \tag{5.95}
$$

Equations (5.91-5.95) are the off-diagonal densities correct to leading (first) order in gradients. We are now ready to compute the contribution from the off-diagonal densities in the kinetic equation (5.86). Making use of the hermiticity properties $\Delta^*_{z12} = \Delta_{z21}$, $\Sigma^*_{z12} = \Sigma_{z21}$, $g^{s\,*}_{ad12} = g^s_{ad21}$, and (5.91-5.95), after some algebra one finds that the contribution from the off-diagonal densities vanishes:

$$
\begin{aligned}
\Sigma_{z12}g^s_{3d21} - \Sigma_{z21}g^s_{3d12} + \Delta_{z12}g^s_{0d21} - \Delta_{z21}g^s_{0d12} &= 0 \\
(D_z m_{hd})_{12}\partial_{k_z}g^s_{1d21} + (D_z m_{hd})_{21}\partial_{k_z}g^s_{1d12} + (D_z m_{ad})_{12}\partial_{k_z}g^s_{2d21} + (D_z m_{ad})_{21}\partial_{k_z}g^s_{2d12} &= 0\,,
\end{aligned} \tag{5.96}
$$

where we neglected the collision terms that are suppressed by derivatives. With the help of this remarkable result and the constraint equations (5.65-5.67) the kinetic equation (5.86) finally reduces to the familiar form [52, 53]

$$
\frac{1}{\tilde{k}_0}\Big[k\cdot\partial - \frac{1}{2}\Big(\partial_z|m_d|^2_i - \frac{s}{\tilde{k}_0}\partial_z\,\big[|m_d|^2_i(\partial_z\theta_{di} + 2\Delta_{z\mathtt{ii}})\big]\Big)\partial_{k_z}\Big]g^s_{0d\mathtt{ii}} = \mathcal{K}^s_{0d\mathtt{ii}}\,. \tag{5.97}
$$

Note that this equation holds for both $g^{s<}_{0d}$ and $g^{s>}_{0d}$. Recall furthermore that for spin conservation the space-time transients are constrained to satisfy $g^s_{0d\mathtt{ii}} = g^s_{0d\mathtt{ii}}(k, t - \vec{v}_\parallel\cdot\vec{x}_\parallel, z)$. The collisional contributions from the constraint equations are higher order in gradients and can be consistently neglected.

The functions g^s_{ad} $(a = 0,1,2,3)$ satisfy four apparently different equations (5.80–5.83). These functions are related by the constraint equations (5.56–5.59), which reduce the number of independent functions to a single one, g^s_{0d}, projected on the quasiparticle shell (5.68). We will show in an appendix in Paper II that all four kinetic equations are actually mutually dependent in a self-consistent manner, and thus equivalent to the kinetic equation for g^s_{0d}, *plus* the on-shell equation (5.68). Our proof includes not only the flow term, but also the collisional sources. The equivalence of these equations including the collision term is a very nontrivial consistency check of our approach to the kinetics of fermions.

5.3. Boltzmann transport equation for CP-violating fermionic densities

Since our primary interest is transport of CP-violating densities, we show here how – starting with the Kadanoff-Baym equations for fermions (2.55) (*cf.* also (5.97)) – one obtains the on-shell Boltzmann

transport equations. The crucial difference with respect to the scalar case discussed in section 4.1 is the CP-violating *semiclassical force* in the flow term of the kinetic equation for fermions. The force appears at second order in gradients, or equivalently at first order in an expansion in $\hbar$.

We begin our analysis with recalling the C and CP transformations for the fermionic fields,

$$\begin{aligned} \psi^c(u) &\equiv \mathcal{C}\psi(u)\mathcal{C}^\dagger = i\gamma^0\gamma^2\bar\psi^T(u), \\ \bar\psi^c(u) &\equiv \mathcal{C}\bar\psi(u)\mathcal{C}^\dagger = \psi^T(u)i\gamma^0\gamma^2\,, \end{aligned} \tag{5.98}$$

and for Dirac's γ-matrices,

$$\begin{aligned} \mathcal{C}\gamma^{\mu T}\mathcal{C}^\dagger &= i\gamma^0\gamma^2\gamma^{\mu T}(i\gamma^0\gamma^2)^\dagger = -\gamma^\mu \\ \mathcal{C}\gamma^5\mathcal{C}^\dagger &= i\gamma^0\gamma^2\gamma^5(i\gamma^0\gamma^2)^\dagger = \gamma^5\,. \end{aligned} \tag{5.99}$$

In the relativistic limit the Standard Model fermions are chiral and couple differently to the weak interactions, so that both charge C and parity P symmetries are strongly violated. But the combined symmetry CP is violated only very weakly, hence it is natural to consider antiparticles to be related to particles by a CP transformation. Therefore we also quote the parity transformations,

$$\begin{aligned} \psi^p(u) &\equiv \mathcal{P}\psi(u)\mathcal{P}^\dagger = \gamma^0\psi(\bar u), \qquad \mathcal{P}\gamma^\mu\mathcal{P}^\dagger = \gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger} = \gamma_\mu \\ \bar\psi^p(u) &\equiv \mathcal{P}\bar\psi(u)\mathcal{P}^\dagger = \bar\psi(\bar u)\gamma^0\,, \qquad \mathcal{P}\gamma^5\mathcal{P}^\dagger = \gamma^0\gamma^5\gamma^0 = -\gamma^5\,. \end{aligned} \tag{5.100}$$

Combining these relations with the definitions of the fermionic Wightman functions (2.5), we easily find the relations for their C- and CP-transformations:

$$S^<(u,v) \xrightarrow{\mathcal{C}} \gamma^0\gamma^2 S^{>T}(v,u)\gamma^0\gamma^2 \tag{5.101}$$

$$S^<(u,v) \xrightarrow{\mathcal{CP}} -\gamma^2 S^{>T}(\bar v,\bar u)\gamma^2\,, \tag{5.102}$$

which in the Wigner representation become

$$S^<(k,x) \xrightarrow{\mathcal{C}} \gamma^0\gamma^2 S^{>T}(-k,x)\gamma^0\gamma^2 \equiv S^{c<}(k,x) \tag{5.103}$$

$$S^<(k,x) \xrightarrow{\mathcal{CP}} -\gamma^2 S^{>T}(-\bar k,\bar x)\gamma^2 \quad \equiv S^{cp<}(\bar k,\bar x)\,; \tag{5.104}$$

analogous relations hold for $S^{c>}$ and $S^{cp>}$. As we did in the scalar case in (4.24–4.25), we employ an additional inversion of the spatial parts of position $\bar x^\mu = (x^0, -x^i)$ and momentum $\bar k^\mu = (k^0, -k^i)$ in the definition of S^{cp}. In the case of several mixing fermions, the Wigner function is a matrix in flavor space which is affected by the transposition, too.

We have argued in section 3 that a weak coupling reduction to the on-shell limit of the equation of motion (2.55) for the fermionic Wigner function results in the equation

$$\Big(\not k + \frac{i}{2}\not\partial_x - \big(m_h(x) + i\gamma^5 m_a(x)\big)\,\mathrm{e}^{-\frac{i}{2}\overleftarrow{\partial}_x\cdot\partial_k}\Big)S^{>,<}(k,x) = \mathcal{C}_\psi(k,x)\,. \tag{5.105}$$

Taking account of the hermiticity property (2.55), hermitean conjugation and transposition lead to

$$\Big(\not k^T - \frac{i}{2}\not\partial_x^T - \big(m_h^*(x) + i\gamma^5 m_a^*(x)\big)\,\mathrm{e}^{\frac{i}{2}\overleftarrow{\partial}_x\cdot\partial_k}\Big)S^{>,<}(k,x)^T = -\gamma^0\mathcal{C}_\psi^*(k,x)\gamma^0\,. \tag{5.106}$$

By commuting $\gamma^0\gamma^2$ through and reversing the sign of the 4-momentum, we get the following equation for the C-conjugate of the Wigner function (5.103),

$$\Big(\not{k} + \frac{i}{2}\,\not{\partial}_x - \big(m_h^*(x) + i\gamma^5 m_a^*(x)\big)\,\mathrm{e}^{-\frac{i}{2}\overleftarrow{\partial}_x\cdot\partial_k}\Big) S^{c<,>}(k,x) = \gamma^2\mathcal{C}_\psi^*(-k,x)\gamma^2\,. \tag{5.107}$$

Similarly, by commuting γ^2 through (5.106), we arrive at the equation for the CP-conjugate Wigner function (5.104):

$$\Big(\bar{\not{k}} + \frac{i}{2}\,\not{\partial}_{\bar{x}} - \big(m_h^*(x) - i\gamma^5 m_a^*(x)\big)\,\mathrm{e}^{-\frac{i}{2}\overleftarrow{\partial}_{\bar{x}}\cdot\partial_{\bar{k}}}\Big) S^{cp<,>}(k,x) = -\gamma^0\gamma^2\mathcal{C}_\psi^*(-k,x)\gamma^0\gamma^2\,. \tag{5.108}$$

Several remarks are now in order. Note first that the presence of imaginary elements in the mass matrices, $m_h \equiv (1/2)(m + m^\dagger)$ and $m_a \equiv (1/2i)(m - m^\dagger)$, may violate both C and CP symmetry. This is true, of course, provided the imaginary parts cannot be removed by field redefinitions. Second, while C is not in general violated by a (real) antihermitean mass term m_a, CP is violated, provided $m_a = m_a(x)$ is space-time dependent, such that it cannot be removed by field redefinitions [81]. Finally, Eqs. (5.107-5.108) illustrate where in the collision term are potential sources of C and CP-violation. As expected, both C and CP may be violated in the presence of complex Yukawa couplings, appearing in collision or mass terms. This can be seen from the complex conjugation of the collision term, $\mathcal{C}_\psi^*$. Further, the weak interaction vertices contain $P_{L,R}$, which is also a property of our model Lagrangean (2.14). Now since $\gamma^2 P_{L,R}\gamma^2 = P_{R,L}$ flips chirality, while $(\gamma^0\gamma^2)P_{L,R}(\gamma^0\gamma^2) = P_{L,R}$ leaves it invariant, we conclude that the weak interaction vertices violate C, but they are invariant under CP, as they should. Using the explicit expressions for the collision terms in Paper II and (5.1.2), one can show that the right hand side of equation (5.108) is equal to $\mathcal{C}_\psi$, with all Wigner functions, both scalar and fermionic, replaced by their CP-conjugate counterparts, and the Yukawa coupling matrices y replaced by y^*. Further details of the analysis of the collision term are left for Paper II.

A careful look at equation (5.108) shows that the kinetic operator for $S^{cp}(k)$ commutes with $P_s(\bar{k})$ rather than with $P_s(k)$. So the spin-diagonal ansatz for the CP-conjugate Wigner function, already written in the mass diagonal basis, is

$$S^{cp}_{\mathtt{dii}}(k,x) = \sum_s S^{cp}_{s\mathtt{dii}}(k,x) = \sum_s iP_s(\bar{k})\big[s\gamma^3\gamma^5 g^{cps}_{0\mathtt{dii}} - s\gamma^3 g^{cps}_{3\mathtt{dii}} + \mathbb{1} g^{cps}_{1\mathtt{dii}} - i\gamma^5 g^{cps}_{2\mathtt{dii}}\big](k,x)\,. \tag{5.109}$$

Now there are several ways to obtain the constraint and kinetic equation for the component function $g^{cps}_{0\mathtt{dii}}$. First, one could repeat all the steps from the previous section. Since the mass term in the CP-conjugate equation (5.108) is the complex conjugate of the mass term appearing in the equation for the original Wigner function (5.105), we would begin by carrying out the flavor transformation, this time using the rotation matrices $\mathbf{X}^*$ and $\mathbf{Y}^*$, then taking the appropriate spinorial traces. The anti-hermitean and hermitean parts would lead to constraint and kinetic equations for the CP-conjugate component functions, respectively, and finally to the desired equations for $g^{cps}_{0\mathtt{dii}}$.
Another possibility is to note that equation (5.108) for S^{cp} is obtained from equation (5.105) for S by replacing the mass by its complex conjugate and replacing all explicit occurrences of the momentum k by $\bar{k}$, as well as sending all spatial derivatives ∂_x to $\partial_{\bar{x}}$, and finally replacing the collision term on the right hand side, as indicated in Eq. (5.108). When we apply exactly the same changes to the constraint and the kinetic equation for g_{0d}, which includes $\theta \to -\theta$ and $\Delta_z \to -\Delta_z$ because of the complex conjugated mass, we find the respective equations for g^{cp}_{0d}.

The simplest method, however, is based on the observation that, because of relation (5.104), the Wigner function S contains complete information about its CP-conjugate S^{cp}, and the same then holds for the component functions g_0 and g_0^{cp} as well. Relation (5.104) is covariant under the flavor rotation,

$$S_d^{cp<}(k,x) \equiv \mathbf{Y}^* S^{cp<}(k,x)\mathbf{X}^T = -\gamma^2\left(\mathbf{Y}S^{>}(-k,x)\mathbf{X}^\dagger\right)^T\gamma^2 = -\gamma^2 S_d^{>T}(-k,x)\gamma^2\,, \tag{5.110}$$

so we have to match the decomposition (5.109) for $S^{cp<}$ with

$$-\gamma^2[S^{>}_{d\mathtt{ii}}(-k,x)]^T\gamma^2 = \sum_s iP_s(-\bar{k})\big[-s\gamma^3\gamma^5 g^{>-s}_{0d\mathtt{ii}} + s\gamma^3 g^{>-s}_{3d\mathtt{ii}} + \mathbb{1}g^{>-s}_{1d\mathtt{ii}} + i\gamma^5 g^{>-s}_{2d\mathtt{ii}}\big](-k,x)\,. \tag{5.111}$$

This implies the identification

$$g^{cps<}_{0d\mathtt{ii}}(k,x) = -g^{>-s}_{0d\mathtt{ii}}(-k,x)\,, \tag{5.112}$$

where we used $P_s(-k) = P_s(k)$. Looking at the constraint equation (5.68) or the spectral ansatz (5.76) for $g^{>s}_{0d\mathtt{ii}}$, we immediately see that the component functions of the CP-conjugate Wigner function live on the same energy shells ω_{si} as the ones of the original Wigner function. In other words, the dispersion relation for antifermions is identical to that for fermions, a conclusion that can also be reached by considering the CP-conjugated constraint equations. Therefore we can make the spectral ansatz

$$g^{<cps}_{0d\mathtt{ii}}(k,x) = 2\pi\sum_{\pm}\frac{\delta(k_0\mp\omega_{\pm si})}{2\omega_{\pm si}Z_{\pm si}}\,|\tilde{k}_0|\,n^{cp}_{si}(k,x)\,. \tag{5.113}$$

Inserting this and the spectral ansatz (5.76) for $g_0^>$ into (5.112) shows that the particle density at negative energies is related to the density of antiparticles. For k_0 on the positive or negative energy shell we have

$$n^{cp}_{si}(k_0,\vec{k},x) = 1 - n_{-si}\big(-k_0,-\vec{k},x\big)\,. \tag{5.114}$$

In order to find the kinetic equation for the CP-conjugate component function, let us first recall the kinetic equation (5.97) for $g^s_{0d\mathtt{ii}}$ from section 5.2.5:

$$\frac{1}{\tilde{k}_0}\Big[k\cdot\partial - \frac{1}{2}\Big(\partial_z|m_d|^2_i - \frac{s}{\tilde{k}_0}\partial_z\left[|m_d|^2_i(\partial_z\theta_{di}+2\Delta_{z\mathtt{ii}})\right]\Big)\partial_{k_z}\Big]g^s_{0d\mathtt{ii}}(k,x) = \mathcal{K}^s_{0d\mathtt{ii}}(k,x)\,. \tag{5.115}$$

Again we make use of relation (5.112) and find

$$\frac{1}{\tilde{k}_0}\Big[k\cdot\partial - \frac{1}{2}\Big(\partial_z|m_d|^2_i - \frac{s}{\tilde{k}_0}\partial_z\left[|m_d|^2_i(\partial_z\theta_{di}+2\Delta_{z\mathtt{ii}})\right]\Big)\partial_{k_z}\Big]g^{cps}_{0d\mathtt{ii}}(k,x) = -\mathcal{K}^{-s}_{0d\mathtt{ii}}(-k,x)\,. \tag{5.116}$$

Needless to say, all ways of obtaining these equations yield the same results, although this might not be obvious for the collision term. Starting with the right hand side of equation (5.108), one would expect to find

$$\mathcal{K}^{cps}_{0d\mathtt{ii}}(k,x) \equiv \frac{1}{2}\mathrm{Tr}\left(\mathbb{1}P_s(\bar{k})\gamma^0\gamma^2\mathcal{C}^*_{\psi d_{\mathtt{ii}}}(-k,x)\gamma^0\gamma^2 + \text{h.c.}\right) \tag{5.117}$$

as the collisional contribution to the kinetic equation for g_0^{cp}. But since this is a trace in spinor space and we are looking at the diagonal elements in flavor space, we can apply a transposition to find

$$\begin{aligned}
\mathcal{K}^{cps}_{0d\mathtt{ii}}(k,x) &= \frac{1}{2}\mathrm{Tr}\left(\left(\gamma^0\gamma^2P_s(\bar{k})\gamma^0\gamma^2\right)^T\mathcal{C}^\dagger_{\psi d_{\mathtt{ii}}}(-k,x) + \text{h.c.}\right)\\
&= \frac{1}{2}\mathrm{Tr}\left(P_{-s}(k)\mathcal{C}_{\psi d_{\mathtt{ii}}}(-k,x) + P^\dagger_{-s}(k)\mathcal{C}_{\psi d}{}^\dagger_{\mathtt{ii}}(-k,x)\right) = -\mathcal{K}^{-s}_{0d\mathtt{ii}}(-k,x)\,.
\end{aligned} \tag{5.118}$$

We define the distribution function for particles as the projection of the phase space densities on the positive mass shell,

$$f_{si+}(\vec{k},x) \equiv n_{si}(\omega_{si},\vec{k},x)\,, \tag{5.119}$$

and then the kinetic equation for g_0^{cp} suggests to define the distribution function for antiparticles as

$$f_{si-}(\vec{k},x) \equiv n_{si}^{cp}(\omega_{si},\vec{k},x) = 1 - n_{-si}(-\omega_{si},-\vec{k},x)\,. \tag{5.120}$$

We will verify in Paper II that $f_{si\pm}(\vec{k})$ indeed measure the density of particles and antiparticles with momentum $\vec{k}$ and spin s. To obtain the Boltzmann equations for these densities we have to integrate (5.115) and (5.116) over positive frequencies:

$$\frac{1}{Z_{si}}\big(\partial_t + \vec{v}_{si}\cdot\nabla_{\vec{x}} + F_{si}\partial_{k_z}\big)f_{si+} = \int_0^\infty \frac{dk_0}{\pi}\mathcal{K}^s_{0d\,\mathtt{ii}}(k,x) \tag{5.121}$$

$$\frac{1}{Z_{si}}\big(\partial_t + \vec{v}_{si}\cdot\nabla_{\vec{x}} + F_{si}\partial_{k_z}\big)f_{si-} = \int_0^\infty \frac{dk_0}{\pi}\mathcal{K}^{cps}_{0d\;\mathtt{ii}}(k,x)\,, \tag{5.122}$$

with the quasiparticle velocity and semiclassical force

$$\vec{v}_{si} = \frac{\vec{k}}{\omega_{si}} \tag{5.123}$$

$$F_{si} = -\frac{1}{2\omega_{si}}\Big(\partial_z|m_d|_i^2 - \frac{s}{\tilde{\omega}_{si}}\partial_z\left[|m_d|_i^2(\partial_z\theta_{di} + 2\Delta_{z\mathtt{ii}})\right]\Big)\,. \tag{5.124}$$

Note that, in addition to the classical terms, both $\vec{v}_{si}$ and F_{si} contain spin dependent contributions arising from derivatives of the pseudoscalar mass term, and thus a potential for inducing CP-violating effects.
The Boltzmann equation (5.122) for the antiparticle densities can also be obtained by integrating the kinetic equation for g_0 over negative energies, and then using (5.120). This procedure has often been advocated as a way of identifying CP-violating densities. It works, because the complete information about $S^{<cp}$ is contained in $S^{>}$, which in the on-shell limit is completely determined by $S^{<}$, as can be shown with the help of the fermionic spectral sum rule (2.60).

For practical baryogenesis calculations one often deals with problems close to thermal equilibrium, in which case it is convenient to treat the Boltzmann equations (5.121) and (5.122) in linear response approximation with respect to deviations from thermal equilibrium. We note that the thermal equilibrium distribution function in the wall frame has the form

$$f_{si}^{\rm eq} = \frac{1}{e^{\beta\gamma_w(\omega_{si}+v_wk_z)}+1}, \tag{5.125}$$

with $\beta = 1/T$, $\vec{v}_w = v_w\hat{z}$ is the wall velocity and $\gamma_w = (1-v_w^2)^{-1/2}$ is the corresponding boost factor. This is a generalization of the thermal equilibrium distribution function (2.73) which includes both the effects of CP-violation from a pseudoscalar mass and of a moving plasma on thermal equilibrium. We show in Paper II that this form for the equilibrium distribution function leads to the correct energy conservation in the collision term. Furthermore, the distribution function (5.125) satisfies

$$Z_{si}^{-1}\big(\partial_t + \vec{v}_{si}\cdot\nabla_{\vec{x}} + F_{si}\partial_{k_z}\big)f_{si}^{\rm eq} = -v_wf_{si}^{\rm eq}(1-f_{si}^{\rm eq})Z_{si}^{-1}\beta F_{si} \tag{5.126}$$

such that, for a wall at rest, Eq. (5.125) satisfies the flow equation exactly, as one would expect from the correct equilibrium distribution function. Note that the degeneracy between the states with opposite spin is broken by this equilibrium, because particles with opposite spin satisfy different energy-momentum dispersion relations.

Our main interest is the transport equation for a CP-violating distribution function that can eventually lead to the creation of a Baryon asymmetry. We first define

$$f_{si\pm} = f_{si}^{\rm eq} + \frac{1}{2}\delta f_i^{\rm even} + \delta f_{si\pm}\,, \tag{5.127}$$

where $\delta f_{si\pm}$ are spin dependent correction functions. Since spin dependence occurs only at order $\hbar$ in our Boltzmann equations and all sources are wall velocity suppressed, as for example can be seen in (5.126), we find $\delta f_{si\pm} = O(v_w\hbar)$. The distributions $f_{si\pm}$ can also contain a CP-even deviation from thermal equilibrium, $\delta f_i^{\rm even}$, which is relevant for the dynamics of phase interfaces [40, 74, 75]. Note that, since $\delta f_i^{\rm even}$ is not suppressed by powers of $\hbar$, it does not in general decouple from the equation for the densities $\delta f_{si\pm}$. In order to properly treat the dynamics of CP-violating densities, one has to solve also for the dynamics of CP-conserving densities! Together with $\delta f_i^{\rm even} = O(v_w\hbar^0)$, Eq. (5.126) implies that it suffices to solve the equation for $\delta f_i^{\rm even}$ to leading (classical) order in gradients, which is obtained by adding (5.121) and (5.122),

$$\big(\partial_t + \vec{v}_{0i}\cdot\nabla_{\vec{x}} + F_{0i}\partial_{k_z}\big)\delta f_i^{\rm even} + \beta f_0(1-f_0)v_w\frac{\partial_z|m_d|_i^2}{\omega_{0i}} = \int_0^\infty \frac{dk_0}{\pi}\left(\mathcal{K}^s_{0d\mathtt{ii}}(k) + \mathcal{K}^{cps}_{0d\mathtt{ii}}(k)\right)\,, \tag{5.128}$$

where

$$\vec{v}_{0i} = \frac{\vec{k}}{\omega_{0i}}\,,\qquad F_{0i} = -\frac{\partial_z|m_d|_i^2}{2\omega_{0i}} \tag{5.129}$$

are the classical particle velocity and the classical force, respectively, and $f_{0i} \equiv 1/(e^{\beta\omega_{0i}}+1)$.

Based on the first order correction $\delta f_{si\pm}$, we can form two CP-violating distribution functions:

$$\delta f_{si}^v \;\equiv\; \delta f_{si+} - \delta f_{si-} \tag{5.130}$$

$$\delta f_{si}^a \;\equiv\; \delta f_{si+} - \delta f_{-si-}\,. \tag{5.131}$$

As we will explain in Paper II, δf_{si}^v and δf_{si}^a are related to the vector and axial vector density in phase space, respectively. Working to second order in gradients and to linear order in v_w, we obtain the equation for δf_{si}^v by subtracting (5.122) from (5.121):

$$\big(\partial_t + \vec{v}_{0i}\cdot\nabla_{\vec{x}} + F_{0i}\partial_{k_z}\big)\delta f_{si}^v = \int_0^\infty \frac{dk_0}{\pi}\left(\mathcal{K}^s_{0d\mathtt{ii}}(k) - \mathcal{K}^{cps}_{0d\mathtt{ii}}(k)\right)\,. \tag{5.132}$$

Note that the flow term contains *no source* whatsoever. This means that a CP-violating semiclassical force (5.124) cannot create vector charge densities (this remains no longer true, however, when one includes the possibility of coherent particle production, which can be exacted by a dynamical tracing of flavor mixing [76]).

The transport equation for δf_{si}^a is obtained by changing the sign of s in (5.122) and then subtracting it from (5.121). With the help of (5.126) we find

$$\big(\partial_t + \vec{v}_{0i}\cdot\nabla_{\vec{x}} + F_{0i}\partial_{k_z}\big)\delta f_{si}^a + s\mathcal{S}_i^{\rm flow} = \int_0^\infty \frac{dk_0}{\pi}\left(\mathcal{K}^s_{0d\mathtt{ii}}(k) - \mathcal{K}^{cp-s}_{0d\mathtt{ii}}(k)\right)\,. \tag{5.133}$$

This equation now has a source:

$$\mathcal{S}_i^{\rm flow} \;=\; -\frac{1}{2}v_w f_{0i}(1-f_{0i})\beta\Big\{\delta F_i + F_{0i}\frac{\delta\omega_i}{\omega_{0i}} + F_{0i}\delta\omega_i\beta(1-2f_{0i}) + F_{0i}\delta Z_i\Big\} \tag{5.134}$$

$$+\frac{1}{2}\Big\{\delta Z_i\big(\partial_t + \vec v_{0i}\cdot\nabla_{\vec x} + F_{0i}\partial_{k_z}\big) + \frac{\delta\omega_i}{\omega_{0i}}\vec v_{0i}\cdot\nabla_{\vec x} + \Big[F_{0i}\frac{\delta\omega_i}{\omega_{0i}} + \delta F_i\Big]\partial_{k_z}\Big\}\delta f_i^{\rm even}\,, \tag{5.135}$$

where the definitions

$$\delta F_i \;\equiv\; \frac{\partial_z[|m_d|_i^2(\partial_z\theta_{di} + 2\Delta_{z\tt ii})]}{\omega_{0i}\tilde\omega_{0i}}$$

$$\delta\omega_i \;\equiv\; \frac{[|m_d|_i^2(\partial_z\theta_{di} + 2\Delta_{z\tt ii})]}{\omega_{0i}\tilde\omega_{0i}} \quad , \quad \delta Z_i \equiv \frac{\omega_{0i}}{\tilde\omega_{0i}^2}\,\delta\omega_i \tag{5.136}$$

have been used. We now pause to comment on the physical meaning of the various terms appearing in the Boltzmann transport equation (5.133). First, the CP-violating density, δf_{si}^a, evolves on phase space according to the standard Boltzmann flow derivative, $d/dt \equiv \partial_t + \vec v_{0i}\cdot\nabla_{\vec x} + F_{0i}\partial_{k_z}$. Second, various CP-violating sources are displayed in (5.134). The first term, δF_i, is the source arising from the semiclassical force that has been usually taken into account in the WKB-literature [35, 39]. Note that the second term has a similar origin. Indeed, it arises from the CP-violating deviation in the energy-momentum relation appearing in the force term, $-(\partial_z|m_d|_i)/2\omega_{si}$. The third term originates from the CP-violating split in the dispersion relation in the equilibrium solution (5.125), and it thus vaguely resembles a local version of spontaneous baryogenesis. And finally, the fourth term arises from the gradient "renormalization" of the Wigner function. All four source terms are of the same order in gradients, and hence, *a priori*, they are all equally important. In Paper II we take moments of the Boltzmann equation (5.133) in order to obtain fluid equations. Since the source term (5.134) is symmetric in momentum, it will contribute to the zeroth moment equation. The parametric dependence of this source becomes explicit by rewriting it like

$$\int\frac{d^3k}{(2\pi)^3}\mathcal{S}_i^{\rm flow} \;=\; v_w\Big(-\partial_z[|m_d|_i^2(\partial_z\theta_{di}+2\Delta_{z\tt ii})]\frac{\mathcal{I}_a}{(2\pi)^2} + \beta^2\left(\partial_z|m_d|_i^2\right)[|m_d|_i^2(\partial_z\theta_{di}+2\Delta_{z\tt ii})]\frac{\mathcal{I}_b}{(2\pi)^2}\Big)\,. \tag{5.137}$$

In figure 5 we plot the dimensionless integrals $\mathcal{I}_a$ and $\mathcal{I}_b$ as a function of the mass. When $|m_d|_i \ll T$ ($|m_d|_i \gg T$), the source $\mathcal{I}_a$ ($\mathcal{I}_b$) dominates. The sources are comparable in strength when $|m_d|_i \sim T$. Third, the terms (5.135) that couple the CP-even deviation form equilibrium have never been considered in literature. Nevertheless, they are formally of the same order as the source terms (5.134), and thus cannot be neglected. While the first term in (5.135) can be reexpressed in terms of the sum of the collision terms, as indicated by Eq. (5.128), the latter terms cannot, and represent genuine dynamical source terms, which have so far not been included in baryogenesis calculations. Fourth, the right-hand-side of Eq. (5.133) contains the collision terms, to be considered in detail in Paper II. Finally, we remind the reader that all of the source terms in Eq. (5.133) are second order in derivatives (first order in $\hbar$), and are suppressed by the wall velocity.

5.4. Applications

In this section we have developed a formalism for the treatment of CP-violating effects of space-time dependent scalar and pseudoscalar mass terms in kinetic theory. We have shown how the

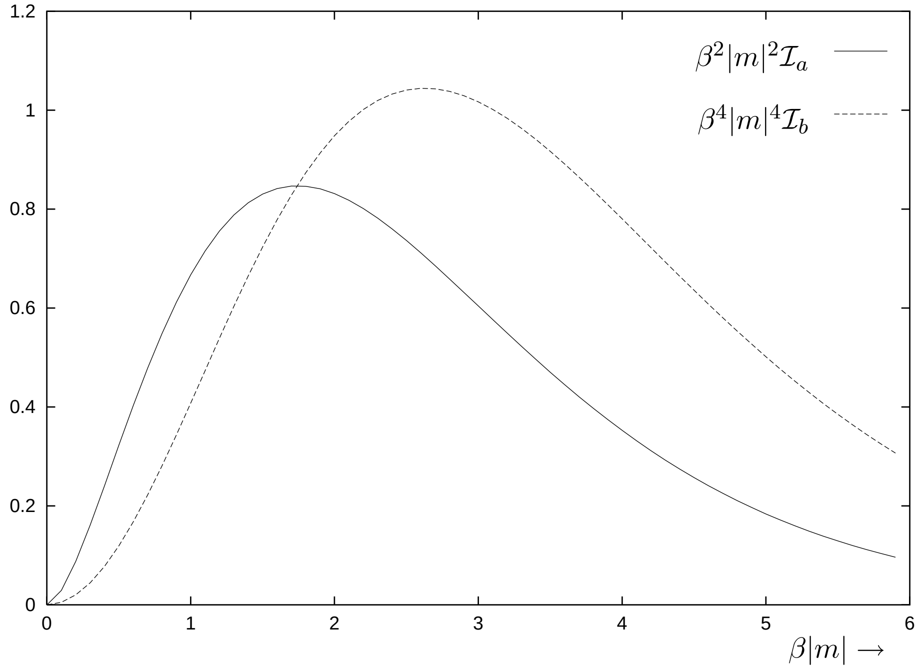

FIG. 5: The integrals $\mathcal{I}_a$ and $\mathcal{I}_b$ in Eq. (5.137) as a function of the mass. We scaled $\mathcal{I}_a$ with $|m|^2$ because this is the way it appears in the source term, analogously $\mathcal{I}_b$ is scaled with $|m|^4$. Note that these contributions enter the flow source with different signs.

effect of a CP-violating shift in the dispersion relation induces an order $\hbar$ semiclassical force in the flow term of the Boltzmann transport equation. The force acts on particles and antiparticles of opposite spin in opposite directions. Since we have included the possibility of fermionic mixing, this formalism is suitable for studies of baryogenesis in supersymmetric theories at a strongly first order electroweak phase transition, in which baryon production is typically biased by CP-violating chargino or neutralino currents. Further, our formalism is also suitable for studying baryogenesis problems from coupling of fermions to the bubble wall in two Higgs doublet models. For completeness we shall now discuss in some detail the case of charginos in the Minimal Supersymmetric Standard Model (MSSM) and its nonminimal extension, and finally remark on baryogenesis sources in two Higgs doublet models.

5.4.1. Minimal Supersymmetric Standard Model (MSSM)

Now that we have a general expression for the semiclassical source in the case of mixing fermions, we want to study two explicit examples, which are of relevance for baryogenesis. First we compute the source in the transport equations for the chargino sector of the MSSM. The chargino mass term reads

$$\overline{\psi}_R \, m \, \psi_L + \text{h.c.}\,, \tag{5.138}$$

where $\psi_R = (\tilde{W}_R^+, \tilde{h}_{1,R}^+)^T$ and $\psi_L = (\tilde{W}_L^+, \tilde{h}_{2,L}^+)^T$ are the chiral fields in the basis of winos. The mass matrix is

$$m = \begin{pmatrix} m_2 & gH_2^* \\ gH_1^* & \mu \end{pmatrix}, \tag{5.139}$$

where H_1 and H_2 are the Higgs field vacuum expectation values and μ and m_2 are the soft supersymmetry breaking parameters, which introduce CP-violation [82]. Since for a reasonable choice of parameters there is no transitional CP-violation in the MSSM, we can take the Higgs expectation values H_1 and H_2 to be real [40, 77]. The matrix that diagonalizes $mm^\dagger$ can be parametrized as [39]

$$U = \frac{\sqrt{2}}{\sqrt{\Lambda(\Lambda+\Delta)}} \begin{pmatrix} \frac{1}{2}(\Lambda+\Delta) & a \\ -a^* & \frac{1}{2}(\Lambda+\Delta) \end{pmatrix}, \tag{5.140}$$

where

$$\begin{aligned} a &= g(m_2 H_1 + \mu^* H_2^*) \\ \Delta &= |m_2|^2 - |\mu|^2 + g^2(h_2^2 - h_1^2) \\ \Lambda &= \sqrt{\Delta^2 + 4|a|^2}\,, \end{aligned} \tag{5.141}$$

and $h_i = |H_i|$. The mass eigenvalues of the charginos are given by

$$m_\pm^2 = \frac{1}{2}\left(|m_2|^2 + |\mu|^2 + g^2(h_1^2 + h_2^2)\right) \pm \frac{\Lambda}{2}\,. \tag{5.142}$$

Upon inserting (5.139-5.142) into (5.70), it is straightforward to show that the chargino source term can be recast as

$$\left[|m_d|^2(\partial_z\theta + 2\Delta_z)\right]_\pm = \mp\frac{g^2}{\Lambda}\Im(\mu m_2)\partial_z\,(h_1 h_2)\;. \tag{5.143}$$

The sources figuring in the transport equation written for charginos in (5.134) can be easily reconstructed from equations (5.141) and (5.143). The result (5.143) agrees with the one found by WKB methods in [39]. In [50, 51] however a different dependence on the Higgs fields was obtained.

5.4.2. Nonminimal Supersymmetric Standard Model (NMSSM)

We now consider an extension of the MSSM which contains a singlet field S in the Higgs sector, which can induce additional CP-violation in the Higgs sector. In particular, the Higgs vacuum expectation values may become complex. The singlet couples to higgsinos, and therefore we obtain the mass matrix by generalizing the higgsino-higgsino component of the chargino mass matrix (5.139)

$$\mu \to \tilde{\mu} = \mu + \lambda S\,, \tag{5.144}$$

where λ is the coupling for the higgsino-higgsino-singlet interaction. The field content we consider is the same as in the MSSM, so the mass matrix is

$$m = \begin{pmatrix} m_2 & gH_2^* \\ gH_1^* & \tilde{\mu} \end{pmatrix}. \tag{5.145}$$

This matrix is still diagonalized by U in (5.140). We write the Higgs expectation values as

$$H_i = h_i \mathrm{e}^{i\theta_i} \quad , \quad i = 1, 2\,, \tag{5.146}$$

where only one phase is physical. With the gauge constraint [78]

$$h_1^2 \theta_1' = h_2^2 \theta_2' \tag{5.147}$$

we can write

$$\theta_1' = \frac{h_2^2}{h^2}\theta' \quad , \quad \theta_2' = \frac{h_1^2}{h^2}\theta'\,, \tag{5.148}$$

where $\theta = \theta_1 + \theta_2$ is the physical CP-violating phase, and $h^2 = h_1^2 + h_2^2$. Now everything is prepared to write the NMSSM-source term. It can be divided into three contributions, which have to be added. The first one is a generalization of the chargino source (5.143)

$$\big(|m_d|^2(\partial_z\theta_d + 2\Delta_z)\big)_{h_1 h_2 \pm} = \mp\frac{g^2}{\Lambda}\Im(\tilde{\mu} m_2 \mathrm{e}^{i\theta})(h_1 h_2)' \tag{5.149}$$

for the case involving a new scalar field S and complex Higgs expectation values. In addition to this there are two new types of sources. One of them is proportional to a derivative of the CP-violating phase θ in the Higgs sector:

$$\big(|m_d|^2(\partial_z\theta_d + 2\Delta_z)\big)_{\theta\pm} = -\frac{g^2\theta'}{\Lambda}\left(\Big(\Lambda \pm (|m_2|^2 + |\tilde{\mu}|^2)\Big)\frac{h_1^2 h_2^2}{h^2} \mp \Re(\tilde{\mu} m_2 \mathrm{e}^{i\theta}) h_1 h_2\right). \tag{5.150}$$

Finally, there is a source that can be written as a derivative of the singlet condensate:

$$\begin{aligned}\big(|m_d|^2(\partial_z\theta_d + 2\Delta_z)\big)_{S\pm} \;=\; & \pm\frac{\lambda g^2}{\Lambda}\Im(m_2 H_1 H_2 S') \\ & + \frac{\lambda g^2}{2\Lambda}\Big(\Lambda \pm (|\tilde{\mu}|^2 + g^2 h^2 - |m_2|^2)\Big)\Im(\tilde{\mu}^* S')\,.\end{aligned} \tag{5.151}$$

The mass eigenvalues $m_\pm$, that is the diagonal elements of $|m_d|^2$, can be obtained from the corresponding expression (5.142) in the MSSM part with the replacement $\mu \to \tilde{\mu}$.

6. CONCLUSION

In this work we perform a controlled first principle derivation of transport equations for a model Lagrangean of chiral fermions Yukawa-coupled to a complex scalar. Our treatment is accurate to first order in an $\hbar$ expansion, or, more concretely, to first order in a gradient expansion with respect to a slowly varying scalar background field, which is formally valid when $\hbar\partial \ll \hbar k$. Being valid to order $\hbar$, our treatment allows us to trace the propagation of CP-violating fluxes, which is of relevance, for example, for electroweak baryogenesis at a first order phase transition. We consistently include the collision term, although in this first paper of our series only in a formal way. The actual evaluation of these terms can be found in Paper II.

We address an open question of electroweak scale baryogenesis mediated by mixing fermions, which couple in a CP-violating manner to a propagating bubble wall of a first order phase transition: which basis should be used to model the kinetics of mixing fermions? We show that, if one is limited to a diagonal approximation, the mass eigenbasis is singled out as the only basis in which the diagonal and off-diagonal elements of the distribution function *decouple* at order $\hbar$ in a derivative expansion. Our derivation is valid when there are no nearly degenerate mass eigenvalues, that is when $\hbar k\cdot\partial \ll \delta(m_d^2)$, where $\delta(m_d^2)$ denotes the (minimum) split in the mass eigenvalues. No such claims can be made for the flavor (weak interaction) basis, in which flavor mixing is present already at the classical level $O(\hbar^0)$. This indicates that the use of the flavor basis in transport equations is at best problematic, unless flavor mixing is consistently included. Of course, the final resolution of this problem can only come from a basis independent treatment, which would include the dynamics of both flavor off-diagonal and diagonal CP-violating densities. At the moment no such treatment is available, however.

We also address various other issues, which comprise a proof that the Kadanoff-Baym equation for a single scalar field can be reduced, in a weakly coupled regime, to an on-shell Boltzmann equation, which includes both the self-energy and the collision term, approximated at the same order in a coupling constant expansion. An inclusion of the hermitean self-energies would be desirable, but has not been done yet. These self-energies mix spin and moreover provide an additional source of CP-violation, which we expect to be of a similar strength as the source in the collision term (see Paper II). Further, we demonstrate that no CP-violating source is present at order $\hbar$ in the flow term of the scalar kinetic equation. We then derive a Boltzmann transport equation for the relevant fermionic quasiparticles with a definite spin. We include all of the CP-violating sources arising from the flow term present at the order $\hbar$, which, of course, include the semiclassical force, but also a few, as-of-yet unencountered, sources. One of the new sources is induced by a CP-even deviation from the equilibrium distribution function, which itself is formally of order $\hbar^0$.

Our method represents a formalized and controlled implementation of the original heuristic semi-classical (WKB) treatment of the problem [34, 35], in which one calculates the relevant (fermionic) dipersion relation accurate to order $\hbar$ by the means of a WKB analysis, and then inserts the result in the appropriate kinetic equation. As we have shown here, when applied with due care, the semi-classical approach leads to a correct semiclassical force in the flow term, but it does not permit the treatment of collisional sources, nor does it allow a critical assessment of the quasiparticle picture of the plasma or the implementation of any effects beyond the quasiparticle picture. None of these limitations apply for the approach presented here.

Our work is completed in Paper II. There we discuss the collision term, which in the present paper was formally maintained in all equations, but never evaluated explicitly, and we derive and study a set of fluid equations, which form an approximation to the Boltzmann equations found here.

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APPENDIX A: GRADIENT EXPANSION IN THE OFF-DIAGONAL SCALAR KINETIC EQUATION

We shall now prove that, provided $k \cdot \partial, (\partial \bar{M}_d^2) \cdot \partial_k \ll \delta(M_d^2)$, gradient expansion applies for the off-diagonal equation. Equation (4.16) is of the form

$$\Big(k \cdot \partial + \frac{1}{2}(\partial \bar{M}_d^2) \cdot \partial_k + \frac{i}{2}\delta(M_d^2) - ik \cdot \delta(\Xi)\Big) i\Delta_{12}^{<} = \mathcal{S}_{12}, \tag{A1}$$

where $\mathcal{S}_{12} = \mathcal{S}_{12}[\Delta_d]$ represents the source composed of the diagonal elements. This equation can be solved by using the method of Green functions as follows. Consider for simplicity the following problem

$$\big(k \cdot \partial + \frac{i}{2}\delta(M_d^2) + \epsilon\big) G_\epsilon^r(k; x, x') = \delta^4(x - x'), \tag{A2}$$

which is solved by the retarded Green function

$$G_\epsilon^r(k; x, x') = \frac{1}{k_0}\theta(t - t')\delta^3\Big(\vec{x} - \vec{x}' - \frac{\vec{k}}{k_0}(t - t')\Big) \exp\Big(-\Big[\frac{i}{2}\frac{\delta(M_d^2)}{k_0} + \epsilon\Big](t - t')\Big), \tag{A3}$$

where ϵ represents an (infinitesimal) positive dissipation term. Formally, this solution contains rapid oscillations, since the exponent varies at the scale $1/\hbar$, which characterizes the off-diagonal elements. One may wonder what happened to these intrinsically quantum oscillations in our solution (4.14) and (4.17). In order to answer this question we consider the solution to (A1)

$$\begin{aligned} i\Delta_{12}(k, x) &= \int d^4x' G_\epsilon^r(k; x, x')\mathcal{S}_{12}(k, x') \\ &= \int^t dt' \mathcal{S}_{12}\Big(k, \vec{x} - \frac{\vec{k}}{k_0}(t - t'), t'\Big)\frac{1}{k_0} \exp\Big(-\Big[\frac{i}{2}\frac{\delta(M_d^2)}{k_0} + \epsilon\Big](t - t')\Big). \end{aligned} \tag{A4}$$

The shift in the position, $\delta\vec{x} = \vec{v}(t - t')$, corresponds to the retardation for a particle moving with the velocity $\vec{v} = \vec{k}/k_0$. A similar retardation shift results when the ∂_k-derivative is included in (A1-A2). Since $\mathcal{S}_{12}$ is varying very slowly when compared with the oscillatory term, we can expand around $t' = t$,

$$\mathcal{S}_{12}\Big(k, \vec{x} - \frac{\vec{k}}{k_0}(t - t'), t'\Big) = \mathcal{S}_{12}(k, x) + (t' - t)\Big(\partial_t + \frac{\vec{k}}{k_0} \cdot \nabla\Big)\mathcal{S}_{12}(k, x) + O(\partial^2) \tag{A5}$$

and integrate (A4) to obtain

$$i\Delta_{12}(k, x) = -\frac{2i}{\delta(M_d^2)}\mathcal{S}_{12}(k, x) - \frac{4}{\delta(M_d^2)^2}\, k \cdot \partial\, \mathcal{S}_{12}(k, x) + O(\partial^2). \tag{A6}$$

This converges provided

$$k \cdot \partial \ll \delta(M_d^2), \tag{A7}$$

which is precisely the criterion for the validity of the gradient expansion in the off-diagonal equations.

One may argue that the derivatives acting on the mass must be included as well. To study the role of these derivatives, we assume that both the source and the masses depend on the time variable only. In this case the off-diagonal equation of motion reads

$$\big[k_0\partial_t + \frac{1}{2}(\partial_t \bar{M}_d^2)\partial_{k_0} + \frac{i}{2}\delta(M_d^2) - ik_0\delta(\Xi_t)\big] i\Delta_{12}(k_0, t) = \mathcal{S}_{12}(k_0, t) \tag{A8}$$

for which the retarded Green function equation is

$$\big[k_0\partial_t + \frac{1}{2}(\partial_t \bar{M}_d^2)\partial_{k_0} + \frac{i}{2}\delta(M_d^2) - ik_0\delta(\Xi_t) + \epsilon\big]G^r_\epsilon(k_0,k_0';t,t') = \delta(k_0-k_0')\delta(t-t')\,. \tag{A9}$$

When written in the Fourier space

$$G^r_\epsilon(k_0,k_0';t,t') = \int \frac{d\kappa}{2\pi}e^{i\kappa(t-t')}g^r_\epsilon(k_0,k_0';\kappa) \tag{A10}$$

equation (A9) becomes

$$\big[ik_0\kappa + \frac{1}{2}(\partial_t \bar{M}_d^2)\partial_{k_0} + \frac{i}{2}\delta(M_d^2) - ik_0\delta(\Xi_t) + \epsilon\big]g^r_\epsilon(k_0,k_0';\kappa) = \delta(k_0-k_0') \tag{A11}$$

where we have ignored the higher order gradients in time (that is we took $\partial_t\bar{M}_d^2$ and $\delta(M_d^2)$ to be time independent). The retarded solution of (A11) is given by (the solution proportional to $-\theta(k_0'-k_0)$ corresponds to the advanced Green function),

$$g^r_\epsilon(k_0,k_0';\kappa) = \frac{2}{\partial_t\bar{M}_d^2}\theta(k_0-k_0')\exp\Big(\frac{i}{\partial_t\bar{M}_d^2}\Big[(\kappa-\delta(\Xi_t))({k_0'}^2-k_0^2) + (\delta(M_d^2)-2i\epsilon)(k_0'-k_0)\Big]\Big). \tag{A12}$$

This solution is not unique. Indeed, replacing ${k_0'}^2$ by $k_0'[ak_0'+(1-a)k_0]$ is also a solution, which however differs from (A12) at higher order in gradients, and hence this ambiguity is irrelevant. From (A12) we then have

$$G^r_\epsilon(k_0,k_0';t,t') = \frac{2}{\partial_t\bar{M}_d^2}\theta(k_0-k_0')\delta\Big(t-t'-\frac{k_0^2-{k_0'}^2}{\partial_t\bar{M}_d^2}\Big)\exp\Big(\frac{i(k_0'-k_0)}{\partial_t\bar{M}_d^2}[\delta(M_d^2)-(k_0'+k_0)\delta(\Xi_t)-2i\epsilon]\Big), \tag{A13}$$

such that Eq. (A8) is solved by

$$\begin{aligned} i\Delta_{12}(k,x) &= \int dk_0'dt'G^r_\epsilon(k_0,k_0';t,t')\mathcal{S}_{12}(k_0',t') \qquad \text{(A14)}\\ &= \int_{-\infty}^{k_0} dk_0'\mathcal{S}_{12}\Big(k_0',t-\frac{k_0^2-{k_0'}^2}{\partial_t\bar{M}_d^2}\Big)\frac{2}{\partial_t\bar{M}_d^2}\exp\Big(\frac{i(k_0'-k_0)}{\partial_t\bar{M}_d^2}[\delta(M_d^2)-(k_0'+k_0)\delta(\Xi_t)-2i\epsilon]\Big). \end{aligned}$$

The shift in time $\delta t = -(k_0^2-{k_0'}^2)/(\partial_t\bar{M}_d^2)$ corresponds to the time retardation caused by the force term. Since in the limit of a very slowly varying field ($\partial_t\to 1/T\to 0$), $\mathcal{S}_{12}$ is varying very slowly in comparison with the rapidly oscillating term, we can expand around $t'=t$ and $k_0'=k_0$,

$$\mathcal{S}_{12}\Big(k_0',t-\frac{k_0^2-{k_0'}^2}{\partial_t\bar{M}_d^2}\Big) = \mathcal{S}_{12}(k_0,t) + (k_0'-k_0)\Big(\partial_{k_0}+\frac{k_0'+k_0}{\partial_t\bar{M}_d^2}\partial_t\Big)\mathcal{S}_{12}(k_0,t) + \mathcal{O}(\partial_t^2,\partial_{k_0}^2) \tag{A15}$$

and integrate (A14) to obtain

$$i\Delta_{12}(k_0,t) = -\frac{2i}{\delta(M_d^2)}\Big[1+\frac{2i}{\delta(M_d^2)}\Big(k_0\partial_t + \frac{1}{2}(\partial_t\bar{M}_d^2)\partial_{k_0} - ik_0\delta(\Xi_t)\Big)\Big]\mathcal{S}_{12}(k_0,t) + \mathcal{O}(\partial_t^2). \tag{A16}$$

One of the higher order terms we have dropped is, for example, $4i[(\partial_t\bar{M}_d^2)/(\delta M_d^2)^3]\partial_t\mathcal{S}(k_0,t)$. This converges provided

$$k_0\partial_t, (\partial_t\bar{M}_d^2)\partial_{k_0} \ll \delta(M_d^2). \tag{A17}$$

From this and Eq. (A6-A7) we conclude that the criterion for validity of the gradient expansion in the off-diagonal equations reads

$$k\cdot\partial, (\partial\bar{M}_d^2)\cdot\partial_k, k\cdot\delta(\Xi) \ll \delta(M_d^2). \tag{A18}$$

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[80] Since M_d^2 is a diagonal matrix, the commutator $\frac{i}{2}\left[M_d^2, i\Delta_d^<\right]$ in (4.9), which is formally of order $\hbar^{-1}$, contributes to the off-diagonal equations *via* terms that contain only the off-diagonal elements.

[81] An example where these types of CP-violation can be of relevance for baryogenesis has recently been considered in [76], where a model of *coherent baryogenesis* has been constructed, which is typically operational on grand-unified scales.

[82] We keep the possibility of complex Higgs vacuum expectation values, because we will reuse the formulas in the NMSSM case.